General Purpose Technologies, Specialization, and Output Growth

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Abstract

This paper introduces an economically meaningful concept of generality of an input and defines a *coefficient of technological generality*, an intuitive measure of congruence in the R&D required for improving the quality of various adaptations of a general purpose input. It is shown in a monopolistic competition model with heterogeneous firms that if the proposed coefficient is less than zero, then firms would prefer to outsource the R&D and production of the input to a specialized supplier in the long run, and specialization will lead to faster output growth. If coefficient is greater than zero, output growth is slower under specialization and specialization cannot be a long run equilibrium. Further, if the coefficient is zero then whether or not specialization occurs is governed by considerations that have been put forward in existing models, and specialization has no impact on the rate of output growth.

Keywords: Technological Generality, Specialization, R&D, Economies of Scale.

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1 Introduction

A long line of economic thought has dwelled on the connections between general purpose technologies, specialization, and output growth. Investigations into these connections to date have proceeded by exploring the linkages between any two of these ideas. We provide a theory that connects all three and illustrate our theory using a monopolistic competition model based on Dixit and Stiglitz (1977). The model provides conditions under which diversity in use of an input leads to specialization in R&D and production of the input, and the conditions under which such specialization results in faster output growth.

Smith (1776) first put forward the idea that specialization leads to increases in input productivity and growth of output, and Smith's arguments were expounded more carefully in models like Romer (1987) and Becker and Murphy (1992). Specialization in this line of thought is not related to generality of input use, although the such a relationship has also been postulated for a long time, at least as far back as Stigler (1951).[1] Stigler's intuition about the emergence of specialized suppliers has also been carefully developed in Bresnahan and Gambardella (1998), but neither they nor Stigler examine impact of specialization in general purpose inputs on the growth rate of output. Other models, including Helpman and Trajtenberg (1998), Aghion and Howitt (1998) and Lipsey, Carlaw, and Bekar (2005), examine the impact of general purpose technologies on output growth, but do not assign any role for specialization in the growth process. The long and intervowen literature on general purpose technologies, specialization and output growth, provides a strong motivation for the effort in this paper to link these concepts.

At the heart of the paper is the relationship between economies of scale and output growth, a relationship that has been explored in a long and venerated literature. The early contributions, especially of Alfred Marshall and Allyn Young, focused on economies of scale that arise at the economy wide or industry level, and were explored more formally in Arrow (1962), Uzawa (1965), Chipman (1970), Romer (1986) and Lucas (1988). The economies of scale in the current paper is

[1] Bresnahan and Trajtenberg (1995) coined the term General Purpose Technology. Stigler (1951) referred to specialized suppliers of general purpose inputs as "general specialists".

a variation of the one that is used in the more recent R&D based growth models of Romer (1990), Grossman and Helpman (1991), Aghion and Howitt (1992) and Jones (1995), and arise from the opportunities to make upfront R&D investments that can improve the quality (or equivalently, efficiency or productivity) of an input used in production.[2] Our contribution is to examine the impact of specialization on output growth when such an input has use across diverse firms, but might need to be adapted for use in the different production processes.

For such a general purpose input, a specialized firm that supplies multiple downstream firms might have more incentive to improve the common input than each individual downstream firm. In such a case, specialization would lead to better exploitation of scale economies, hence the connection between generality of input use, specialization and the rate of output growth. We show in a monopolistic competition framework that when all firms use a common input and there are no costs involved in adapting the input to each firm's production process, then the firms would prefer to source the input from a specialized supplier, and output growth would be faster under specialization. Specialized suppliers act as substitutes for joint research consortiums, creating scale effects from specialization.

Yet, one rarely finds such perfect general purpose inputs, whose improvements in quality can on its own improve productivity across diverse firms. What is more common is that some inputs, like semiconductor chips, often benefit from continual advances in basic scientific or engineering methods, which have to be adapted in different ways across multiple using industries to drive productivity improvements. Bresnahan and Trajtenberg (1995) identify such co-invention as a key characteristic of general purpose technologies.

Hence the economies of scale to be reaped from specialization depend not just on the extent of use of the input, but also on congruence in the R&D required for improving the different adaptations of the input. If there are some general concepts that can be used to increase the quality of the different adaptations of the input, then there would presumably be gains from specializing in making improvements to the set of general concepts. This idea was first put forward by Rosen-

[2] Jones (1999) explores some of the features and limitations of models of output growth where the increasing returns to scale stem from R&D opportunities to improve the production process.

berg (1963), in his study of the emergence of specialized suppliers of machine tools in the United States.[3]

We build upon Rosenberg's notion in Section 2 and define a new *coefficient of technological generality*, an intuitive measure of congruence in the R&D required for improving different adaptations of an input. Within the assumptions of our model, we show in Section 3 that for specialization to occur, it is sufficient that the coefficient of technological generality be less than zero. Further, while specialization can occur if the the coefficient is equal to zero, if specialization is to result in faster output growth, it is both necessary and sufficient that the coefficient be less than than zero.

This mechanism described in this paper is complementary to the ones considered in a recent group of papers focused on production networks, that seeks to understand the emergence of key suppliers in the economy.[4] In the models in this group of papers, key suppliers emerge because competition accentuates firm-level differences over time, either in firm productivity as in Oberfield (2017), or because of positional advantage in an existing network as in Carvalho and Voigtlander (2014).

We suggest a reason why, in the first place, suppliers emerge for inputs that are likely to form hubs for production networks, rather than the manufacture of such inputs occurring within the firms that use these inputs. In contrast to the above papers that do not rely on any property of the input per se but rather on the differences across firms, we focus on a key property of the input that ensures that specialized suppliers would emerge for such general purpose inputs.

In the next section we formalize our notion of technological generality of an input. We also give three examples of general purpose inputs - semiconductor chips, machine tools and chemical plant design - where documented evidence suggests that technological generality was central to the emergence of specialized suppliers.

[3]See Section 2.1 for more details.

[4]Inspired by the results in the Carvalho (2014) and Acemoglu, Carvalho, Ozdaglar, and Tahbaz-Salehi (2012) that firm levels shocks are more likely to have macroeconomic impacts if there are key suppliers in the economy who form hubs for production networks, many authors have developed models were such suppliers emerge endogenously.

2 Technological Generality

There are many inputs whose improvement in quality benefits firms in a particular industry, or a group of closely related industries. There are a few other inputs, like steam engines or semiconductor chips, where the benefits span a wide swathe of industries. At the heart of these general purpose inputs is an idea that has applicability in the production of many different products.

Repeated improvements in the idea that using heat to convert water to steam allows mechanical work to be done in a controlled fashion, led to the use of steam engines in many industries - initially in the mining industry to pump water, then in the textile industry to drive looms, and finally in railroads and steamships to transport materials and people. Similarly, improvements in the idea that semiconducting materials can be used to make electronic components for storage, manipulation and communication of information led to the use of semiconductor chips in the production of a wide variety products - from calculators, radios and hearing aids to medical equipment, automobiles, aircrafts, televisions, computers, robots, and more recently to mobile phones and bionic implants.

While both of these ideas required adaptation to meet the requirements of different industries, there were undoubtedly common elements that once understood, could be adapted for use with modifications. We base our idea of technological generality on this notion of reusability of a common idea. Bresnahan (2011) provides an excellent discussion of the connection between general purpose technologies and reuse of ideas.[5]

To give these notions a concrete form, suppose that there is a continuum of firms, $i \in (0, N)$ that use an input in their production. The firms have an opportunity to invest in R&D to increase the quality of the input. Let $R(i, q, z)$ be the R&D cost to firm i of developing an input of quality q and adapting it for use in its own production process, in an external environment captured by the variable z.

Assume that there exists another firm, the supplier, who understands the common processes

[5] Weitzman (1998) develops a model where economic growth is driven by combining existing ideas. The focus of our paper is slightly different in that the emphasis is on finding new uses of a single general purpose input.

involved in increasing the qualities of the different adaptations of the input used by the various firms. Let $R_s(N, q, z)$ be the R&D cost for this supplier for developing quality q input and adapting it across the N different downstream using firms, in an external environment captured by z.

Let $r(N)$ be the R&D cost of the supplier relative to the total R&D cost incurred if each firm had made the innovation and adaptation on its own,[6]

$$r(N) = \frac{R_s(N, q, z)}{\int_{i=0}^{N} R(i, q, z) di} \tag{1}$$

We define the *coefficient of technological generality*, κ, as the elasticity of the relative R&D cost $r(N)$ with respect to N,

$$\kappa = \frac{N r'(N)}{r(N)}. \tag{2}$$

A value of $\kappa < 0$ means that the supplier's cost of doing the R&D for increasing the quality of different adaptations of the input, relative to the total R&D cost incurred when downstream firms do the R&D on their own, decreases as more firms start using the input. This is likely the case if the R&D across the different firms involves some common processes, which can be done once and the common result then adapted across multiple downstream firms. The case with $\kappa = 0$ is one where it does not make a difference whether R&D is done by a common supplier or separately by each downstream firm. If $\kappa > 0$, then the input shows technological divergence.

The notion of technological generality above is different from the catch-all notion of technology spillovers, which is the transmission of useful information across firms. One could think of technological generality as technology spillovers across industries, but the fact that such inter-industry spillovers are useful likely implies that there are some common ideas that can used in improving production in many industries, and it is this property that is primal, making spillovers useful.[7]

[6]We have implicity assumed that the relative R&D cost depends only on N, a more general version would allow it to depend on q and z as well.

[7]The notion of technological generality is also not equivalent to having economies of scope in R&D. It is possible that for an input with $\kappa < 0$ the supplier's cost of doing R&D is higher than the total cost incurred if the R&D were done at individual firms, for a range of values of N. What is important is that the relative R&D cost, $r(N)$, decline as N increases.

A natural question arises in the notion of technological generality formulated above. Who are these specialized suppliers, and why can they understand the common concepts involved in making improvements to the different adaptations of the general purpose input, while others cannot?

2.1 Who are the Specialists?

In the *Wealth of Nations,* Adam Smith wrote insightfully that improvements in machinery are likely to come from two kinds of people.[8] There are those whose occupation gives them the opportunity to use (or make) some machinery and gain the knowledge to improve upon and presumably find other related uses for the machinery, a point of view that was also stressed by Hayek (1945). And then there are scientists and academics, whose technical or educational background often provides the capability to identify the common or complementary principles at play across different settings.[9] Fortunately, Nathan Rosenberg has furnished excellent examples in both of Smith's categories that can throw some light on the identity of specialized suppliers of general purpose inputs.

An example of the former kind can be found in Rosenberg (1963), a study of hardscrabble entrepreneurs who emerged as suppliers of machine tools in the northeastern United States in mid 19th century. The process of cutting metal was a requirement for many emerging industries at that time, like firearms, bicycles and sewing machines. Metal cutting involves a set of separate operations - milling, boring, grinding, planing - each of which historically faced a similar set of requirements in the different using industries.[10] The early metal cutting tools (or machine tools)

[8]"All the improvements in machinery, however, have by no means been the inventions of those who had occasion to use the machines. Many improvements have been made by the ingenuity of the makers of the machines, when to make them became the business of a peculiar trade; and some by that of those who are called philosophers or men of speculation, whose trade it is not to do any thing, but to observe every thing; and who, upon that account, are often capable of combining together the powers of the most distant and dissimilar objects", Smith (1776).

[9]Arora and Gambardella (1994) examine the relative importance of these two mechanisms of innovation over time.

[10]"Moreover, all machines performing such operations confront a similar collection of technical problems, dealing with such matters as power transmission (gearing, belting, shafting), control devices, feed mechanisms, friction reduction, and a broad array of problems connected with the properties of metals (such as ability to withstand stresses and

were developed by machine shops attached to big manufacturers, especially in the textile and armaments industries. Rosenberg narrates many instances of people in such machine shops, who when improving a tool to solve a specific problem in one industry, came upon general principles that could be reused in other industries. Rosenberg argues that the discovery of general, reusable principles was an important reason for the evolution of factory attached machine shops into independent machine tool suppliers serving multiple customers.[11]

Rosenberg (1998) provides an example of the latter kind of specialists - suppliers of chemical plant design services. The turn of the 20th century saw increasing use of chemical processes in many industries, including food-processing, rubber, leather, petroleum refining, glass, paper, cement, and in metallurgical industries like iron, aluminum and steel. A group of academics at the chemical engineering department (and its fore runners) at the Massachusetts Institute of Technology (MIT), realized that the chemical processing in the different industries relies on variations of a subset of standard unit operations like evaporation, distillation, liquefaction, electrolyzation and condensation.[12] These academics at MIT characterized the different unit operations and the adaptations required for their application in different industries. This codification of knowledge, enabled by the reusability of general principles, paved the way for the emergence of Specialized Engineering Firms (SEFs) that provided chemical plant design services to firms in diverse industries that

heat resistance)", Rosenberg (1963), page 423.

[11]"The machine tool industry, then, originated out of a response to the machinery requirements of a succession of particular industries; while still attached to their industries of origin, these establishments undertook to produce machines for diverse other industries, because the technical skills acquired in the industry of origin had direct application to production problems in other industries; and finally, with the continued growth in demand for an increasing array of specialized machines, machine tool production emerged as a separate industry consisting of a large number of firms most of which confined their operations to a narrow range of products- frequently to a single type of machine tool, with minor modifications with respect to size, auxiliary attachments, or components", Rosenberg (1963), pages 420-421.

[12]Rosenberg (1998) details the contribution of the academics at MIT involved in the codification of this knowledge. The concept of unit operations was first formulated by Arthur D. Little, who lectured at MIT from 1893-1916. Other contributors include W. K. Lewis, William Walker and William McAdams (who together published an influential textbook, *The Principles of Chemical Engineering*), Robert Haslam, and Edwin R. Gilliand.

used chemicals.[13]

Finally, both of Smith's mechanisms were at play in the emergence of specialized suppliers of semiconductor chips. Similar to the case of machine tools, the early specialized semiconductor firms where in most cases started up by employees in the semiconductor divisions of leading companies like AT&T and Hughes Aircraft. But most of these founders were also scientists, usually with advanced degrees from leading universities.[14] These scientists were able to leverage both the technical knowledge from their educational backgrounds, and the entrepreneurial knowledge of possible new uses for semiconductor chips that they gained working in the semiconductor divisions of companies, to understand the common technological principles behind the various uses of semiconductor chips.

The common thread in all three examples above is the identification of inputs that were based on general and reusable ideas, which where then taken over by external suppliers who invested in improving the quality of these inputs.

The next section develops the model. In the model, we do not delve into the differences in technical and entrepreneurial capabilities that characterize the specialized suppliers, instead take as given that there are such specialists who can understand the general requirements across industries. We make many simplifying assumptions to bring out sharply the basic forces at work in the model.

[13] Some of the early SEFs were Universal Oil Products (UOP), Chemical Construction Corporation, Chemical Engineering Corporation and Kellog. Arora and Gambardella (1998) provides a history of the evolution of chemical industry, including the role of SEFs.

[14] After the invention of the semiconductor transistor at AT&T's Bell Labs in 1947, a number of leading technology companies set up R&D and manufacturing programs to make semiconductor chips. The leading manufacturers of semiconductor chips in the 1950s were AT&T, General Electric, Radio Corporation of America, Hughes Aircraft, Texas Instruments and Westinghouse (see Tilton (1971) and Kraus (1973)). Almost all of the early semiconductor suppliers could be traced to the the above companies, especially AT&T and Hughes Aircraft (see Hoefler (1968)). Intel and AMD were started by scientists from Fairchild Camera and Instruments, whose semiconductor division was started by scientists from Shockley Semiconductors, which in turn was started by Dr. William Shockley who was one of the inventors of the transistor at AT&T Bell Labs. The semiconductor division at Texas Instruments was started by Dr. Gordon Teal, a scientist who previously worked at AT&T Bell Labs. Transitron, another early semiconductor manufacturer was started by David Bakalar, who had also worked on transistors at AT&T.

We do not endogenize fully the rate of growth of a GPT, instead model advances in the science related to GPTs as an exogenous process, and use the notion suggested in Rosenberg (1998) that such advances reduce the cost of inventive activity in the sectors that use the GPT. Further, to focus on the role of technological generality, we ignore upstream competition and consider the basic case where there is only one potential supplier. To keep the analysis tractable, we adopt a partial equilibrium approach with aid of a quasilinear utility function, and examine the rate of output growth in the section of the economy that can potentially use the general purpose input.

3 Model

There is a continuum of firms, indexed by $i \in (0, \infty)$, each of which can potentially use a general purpose input in their production process. The demand for the goods that use this common input are generated by a consumer with constant elasticity of substitution (CES) preferences over the goods, and a constant flow of income I to allocate between these goods and another numeraire good. To focus on the section of the economy that comprises firms that can use the input, the utility function of the consumer is assumed to be quasilinear, with the demand functions being generated from the utility maximization problem,

$$\begin{aligned}
\max U(Z(t), Y(t)) &= Z(t) + \frac{\alpha}{\alpha - 1} Y(t)^{\frac{\alpha-1}{\alpha}}, \qquad \alpha > 1 \\
s.t \;\; Z(t) &+ \int_{i=0}^{N(t)} p(i,t) y(i,t) = I, \\
Y(t) &= \left(\int_{i=0}^{N(t)} y(i,t)^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}}, \qquad \eta > 1
\end{aligned}$$

where $Z(t)$ is the quantity of numeraire good consumed at time t, $y(i,t)$ is the quantity of good i consumed at time t, $p(i,t)$ is the price of good i at time t, and $N(t)$ is the measure of firms who are using the input at time t. The above utility function leads to the aggregate and firm-level demand functions,

$$Y(t) = P(t)^{-\alpha} \tag{3}$$

$$y(i,t) = Y(t)\left(\frac{p(i,t)}{P(t)}\right)^{-\eta} = p(i,t)^{-\eta}P(t)^{\eta-\alpha}, \tag{4}$$

where $P(t)$ is the industry price index given by $P(t) = \left(\int_0^{N(t)} p(i,t)^{1-\eta}di\right)^{\frac{1}{1-\eta}}$. The consumer allocates expenditure $P(t)^{1-\alpha}$ on the set of good that use the general purpose input, and $I - P(t)^{1-\alpha}$ on the numeraire good. As the prices of goods that use the general purpose input falls, the consumer allocates more and more of the total expenditure towards the goods. We will impose the condition that $\eta > \alpha$, i.e the firm's demand curve is more elastic than the aggregate demand curve for the input.

3.1 Technology

To bring out the intuition behind the theory sharply, we will assume that the only input to production is the general purpose input. The input could be a piece of capital equipment, like a steam engine or a machine tool, or an intermediate component like a semiconductor chip, or a service like chemical plant design.

There is a quality level associated with each unit of the input. A unit of the input of higher quality can produce more output than one of lower quality. Firms are heterogeneous in their ability to use this input. We denote by $a(i)$ the number of units of output that can be produced by firm i using a unit of the input with unit quality. We assume that each firm's ability, $a(i)$, does not vary over time. The production function is,

$$y(i,t) = a(i)q(i,t)K(i,t),$$

where $y(i,t)$ is the quantity of good i that can be produced with $K(i,t)$ units of the input, each of quality $q(i,t)$, by a firm with ability $a(i)$.

The productivity of the input for firm i at time t is $a(i)q(i,t)$. Since $a(i)$ is constant over time, the growth of input productivity for firm i is simply the growth in quality $q(i,t)$ of the input.

Without any loss of generality, we will arrange the firms so that $a(i)$ is decreasing in i. The cost of producing a unit of the input is the same for all qualities of the input, and we normalize this unit production cost to one.[15]

Firm i can increase its quality $q(i,t)$ by investing in R&D. Improvements in basic science that enables improvements in general purpose inputs, like semiconductor chips, are often the result of joint efforts of firms, universities and other research organizations. We model this by assuming that science behind the general purpose technology advances exogenously at a given rate, and the advancements in science make it easier for firms to improve the quality of their input. For every improvement in quality of the input that the firm makes, the firm also has to adapt the new higher quality input to suit its production process, and this adaptation cost might vary across firms.

These aspects of innovation and adaptation are captured with a research cost function for firm i,

$$R(i,q,z) = h(i)\frac{q^{\sigma}}{z} \tag{5}$$

where $R(i,q,z)$ is the R&D cost to firm i to develop and adapt input of quality q. The function $h(i)$ factors in the cost incurred by the firm for adapting the input, and z captures the state of scientific knowledge. The state of scientific knowledge grows exogenously at the rate g,

$$z(t) = e^{gt}.$$

We assume that there exists a supplier who can do the R&D to improve the quality of the input, adapt it to requirements of each downstream firm, and produce the different adaptations. For now, we assume that the R&D cost function of the supplier to develop input of quality q and adapt it for use by N downstream firms is $R_s(N,q,z)$. We expand more on the function R_s in Section 3.3.

We assume that the supplier faces the same unit production cost for the input as the downstream firms, equal to one. Finally, firms can enter the market to use the input and produce a new good, as long as they pay a fixed cost F. This is a one time cost to be paid to start producing using the general purpose input, and firms do not have to pay any such set up cost when they move on to higher quality versions of the input.

[15] If the input is a piece of capital equipment, we assume that the capital equipment depreciates fully instantaneously.

Firms make their pricing, production and R&D investment decisions in a non-cooperative manner, guided by the consideration to maximize profit. The question of whether or not to rely on an outside supplier for an input is, however, not well described by a non-cooperative setting. Downstream buyers of semiconductor chips, like computer or communication equipment manufacturing firms, often jointly influence semiconductor chip manufacturing firms, through trade associations and industry consortia. Hence we look for a coalition-proof equilibrium in our model, and adopt the equilibrium notion suggested in Bernheim, Peleg, and Whinston (1987).

We first derive the output growth rates in the two polar cases, the integrated case where all firms do the R&D and production of the input in-house, and the specialized case where all firms outsource to the supplier, and show that the output growth rate is higher in the specialized case if and only if the coefficient of technological generality, κ, is less than zero. Then we show that these two polar cases are the only two possible equilibria, i.e., partial specialization is not possible. Finally we show that specialization is the unique long run coalition-proof equilibrium if $\kappa < 0$, and specialization cannot be a long run coalition-proof equilibrium with $\kappa > 0$. If κ is equal to zero, then specialization can be a long run coalition-proof equilibrium, but that determination is not affected by market growth, and is simply governed by a familiar comparison in other models of the size of the supplier markup relative to the fixed R&D investment cost. We start with the vertically integrated case in the next section.

3.2 Vertical Integration

At each point in time, firms choose how much to invest in R&D, which in turn decides the quality of the input available for production. They also manufacture the input with the chosen quality, produce the output, and finally sell the output at the price they choose. Firms make their choices to maximize profits, hence their choices should solve the problem,

$$\begin{aligned}
\max_{p(i,t),q(i,t)} \quad & p(i,t)y(i,t) - K(i,t) - R(i,q(i,t),z(t)) \\
s.t \quad & y(i,t) = p(i,t)^{-\eta}P(t)^{\eta-\alpha} \\
& y(i,t) = a(i)q(i,t)K(i,t) \\
& R(i,q(i,t),z(t)) = h(i)\frac{q(i,t)^{\sigma}}{z(t)}.
\end{aligned}$$

We first find each firm's gross profit as a function of quality, and then use that to find the firm's optimal quality choice. Firm i's optimal price to charge in period t is,

$$p(i,t) = \frac{\eta}{\eta-1}\frac{1}{a(i)q(i,t)} = \frac{m_d}{a(i)q(i,t)}, \tag{6}$$

where $m_d = \frac{\eta}{\eta-1}$ is the markup over quality adjusted unit cost, $\frac{1}{a(i)q(i,t)}$. Solving for $y(i,t)$ and $K(i,t)$ using the demand and production functions above, the gross profit made by firm i at time t can be seen to be,

$$\pi(i,t) = p(i,t)y(i,t) - K(i,t) = \frac{1}{\eta}\left(\frac{a(i)q(i,t)}{m_d}\right)^{\eta-1} P(t)^{\eta-\alpha}, \tag{7}$$

Using the solution for $p(i,t)$ in equation (6), the price index $P(t)$ simplifies to,

$$P(t) = m_d\left(\int_{i=0}^{N(t)} (a(i)q(i,t))^{\eta-1}\, di\right)^{\frac{1}{1-\eta}}. \tag{8}$$

The R&D investment problem of the firm can now be simplified to,

$$\max_{q(i,t)} \pi(i,t) - R(i,q(i,t),z(t)) \tag{9}$$

where $R(i,q(i,t),z(t))$ and $\pi(i,t)$ are given in equations (5) and (7) respectively. Note that the continuum assumption on i means that the price index $P(t)$ is not affected by the firm's choice of $q(i,t)$.

It can be seen from the first order condition on $q(i,t)$ that the optimal choice for every firm is to invest a constant fraction, $\frac{\eta-1}{\sigma}$, of its anticipated profit $\pi(i,t)$ into R&D.[16] For notational

[16]We assume that firms have access to capital markets to finance their R&D.

convenience, we define a new variable φ to denote this constant R&D to gross profit ratio, i.e $\varphi = \dfrac{\eta - 1}{\sigma}$. Using the R&D and gross profits functions, the optimal R&D condition can be written out in terms of $q(i, t)$,

$$q(i,t)^{\sigma-(\eta-1)} = \frac{\varphi z(t)}{\eta h(i)} \left(\frac{a(i)}{m_d}\right)^{\eta-1} P(t)^{\eta-\alpha}. \tag{10}$$

The second order condition for the problem is $\sigma > \eta - 1$, which will ensure that research costs rise faster with quality than profits, and the firm invests a positive fraction of its anticipated profits into R&D. Using the expression for the optimal $q(i, t)$ above, the gross profit function can be written as,

$$\pi(i,t) = \left(\left(\frac{P(t)^{\eta-\alpha}}{\eta}\right)^{\sigma} \left(\left(\frac{a(i)}{m_d}\right)^{\sigma} \frac{\varphi z(t)}{h(i)}\right)^{\eta-1}\right)^{\frac{1}{\sigma-(\eta-1)}}. \tag{11}$$

Note that $\pi(i, t)$ is decreasing with i, if $\dfrac{a(i)^{\sigma}}{h(i)}$ is decreasing in i. We will assume that this is the case, and show below that this is an outcome of other assumptions we make.

Entry

A firm will adopt the general purpose technology at time t and start using the general purpose input to produce a new output good, if the benefit to entering at t is greater than the benefit to waiting. A firm's profit at t, and any future profits at other points in time, is unaffected by whether it entered at t or at any other time. Hence firm i will enter in period t if,

$$(1-\varphi)\pi(i,t)\Delta t \geq F - e^{-\rho\Delta t}F$$

where ρ is the discount factor. The left hand side of the equation is the benefit to entering at time t, and the right hand side is the benefit to waiting. Taking the limit of $\Delta t \to 0$, we get

$$(1-\varphi)\pi(i,t) \geq \rho F.$$

Since $\pi(i, t)$ is decreasing with i, there is a cutoff firm each period with ability $a(N(t))$ and quality $q(N(t), t)$, such that all firms with ability greater than $a(N(t))$ would have entered the market by t. Hence the entry condition above should hold with equality for the cutoff firm, i.e,

$$\pi(N(t),t) = \frac{\rho F}{(1-\varphi)} \equiv \frac{f}{(1-\varphi)}, \tag{12}$$

where, for notational convenience we have defined $f \equiv \rho F$, the annuitized value of the entry cost F. Using the expression for profit from equation (7) for this cutoff firm, we can write the entry condition as,

$$\frac{P(t)^{\eta-\alpha}}{\eta}\left(\frac{a(N(t))q(N(t),t)}{m_d}\right)^{\eta-1} = \frac{f}{1-\varphi}. \tag{13}$$

Equilibrium

Since there are no inter-temporal relationships, the equilibrium of the system can be characterized as a set of static equilibria, one for each t. A static equilibrium at t is the set $\left(\{q(i,t)\}_{i=0}^{N(t)}, N(t), P(t)\right)$ such that the optimal R&D conditions (one for each i) in equation (10) and the entry condition in equation (13) are satisfied. If these two conditions are satisfied, then each firm is making pricing, production, R&D and entry decisions that maximizes its profit, given the choices made by the other firms.

The solution to the set of equations would depend on how $a(i)$ and $h(i)$ are related to each other. In general, we would expect these two variables to move together, an adaptation of the basic innovation that delivers a higher productivity increment would also probably require a higher research cost to make the adaptation. If we assume that these effects cancel each other, i.e,

$$a(i)^{\eta-1} = h(i), \tag{14}$$

then we get a tractable way of characterizing the equilibrium, as we show below.[17] Note also, that since $\sigma > \eta - 1$, the assumption that $a(i)^{\eta-1} = h(i)$ guarantees that $\dfrac{a(i)^{\sigma}}{h(i)}$ (which then is equal to $a(i)^{\sigma-(\eta-1)}$) is decreasing in i, which ensures that $\pi(i,t)$ is decreasing with i for every t, which in turn guarantees the existence of a cutoff firm for every t.

From the optimal R&D condition in equation (10) it is easy to see that, if $a(i)^{\eta-1} = h(i)$, then every firm will choose the same quality for any given t. Denote this common quality by $q(t)$, which

[17] In a related paper, Kyun and Pillai (2017), we explore a general model of diffusion of general purpose technologies without the assumption, but confined to the case where all firms are vertically integrated. The diffusion and growth paths are very similar to the ones described here.

can be obtained from equation (10) as,

$$q(t)=\left(\frac{\varphi z(t)}{\eta m_d^{\eta-1}}P(t)^{\eta-\alpha}\right)^{\frac{1}{\sigma-(\eta-1)}}. \tag{15}$$

The price index then simplifies to,

$$P(t)=m_d\left(\int_{i=0}^{N(t)}\left(a(i)q(t)\right)^{\eta-1}di\right)^{\frac{1}{1-\eta}}=\frac{m_d}{A(N(t))^{\frac{1}{\eta-1}}q(t)}. \tag{16}$$

where we have defined $A(N(t))$ as,

$$A(N(t))=\int_{i=0}^{N(t)}a(i)^{\eta-1}di.$$

As defined, $A(N(t))$ is a productivity adjusted measure of downstream firms. For notational convenience, we will simply use $A(N)$ and $a(N)$ where convenient, remembering that N itself is a function of t. With these modifications, the gross profit of each firm in equation (7) becomes,

$$\pi(i,t)=\frac{a(i)^{\eta-1}}{\eta m_d^{\alpha-1}}\frac{q(t)^{\alpha-1}}{A(N)^{\frac{\eta-\alpha}{\eta-1}}}. \tag{17}$$

The entry condition in equation (13) can be written as,

$$\frac{A(N)^{\frac{\eta-\alpha}{\eta-1}}}{a(N)^{\eta-1}}=\frac{(1-\varphi)}{\eta f m_d^{\alpha-1}}q(t)^{\alpha-1}.$$

Note from equations (7) and (16) that $\frac{1}{\eta m_d^{\alpha-1}}$ is the gross profit obtained (and hence $\frac{(1-\varphi)}{\eta m_d^{\alpha-1}}$ the net profit obtained), if there was only one firm with $q=1$ and $a=1$. With this consideration, we define,

$$v=\frac{(1-\varphi)}{\eta m_d^{\alpha-1}}.$$

The entry condition above can be further simplified using the variable v as,

$$\frac{A(N)^{\frac{\eta-\alpha}{\eta-1}}}{a(N)^{\eta-1}}=\frac{v}{f}q(t)^{\alpha-1} \tag{FE}$$

The optimal R&D condition in equation (10) can also be simplified to,

$$q(t)^{\sigma-(\alpha-1)}=z(t)\frac{\varphi v}{(1-\varphi)}\frac{1}{A(N)^{\frac{\eta-\alpha}{\eta-1}}}. \tag{OR}$$

Together the Free Entry Condition (*FE*) and the Optimal R&D condition (*OR*) is system of two equations in two variables, $q(t)$ and $N(t)$. A graphical illustration of the equilibrium is shown in Figure 1.

Note that $A(N)$ is increasing in N and $a(N)$ is decreasing in N, hence the left hand side of FE condition is an increasing function of N. Hence the FE curve is upward sloping, indicating that as scientific knowledge $z(t)$ increases, it leads both to increases in quality $q(t)$ of the input used by firms, and in the entry of new firms leading to an increase in N. On the other hand, as N increases, each firm has less incentive to do R&D, hence OR curve is downward sloping. As time changes from t to $t' > t$, the OR curve shifts to the right, moving the system to a higher value of q and N.

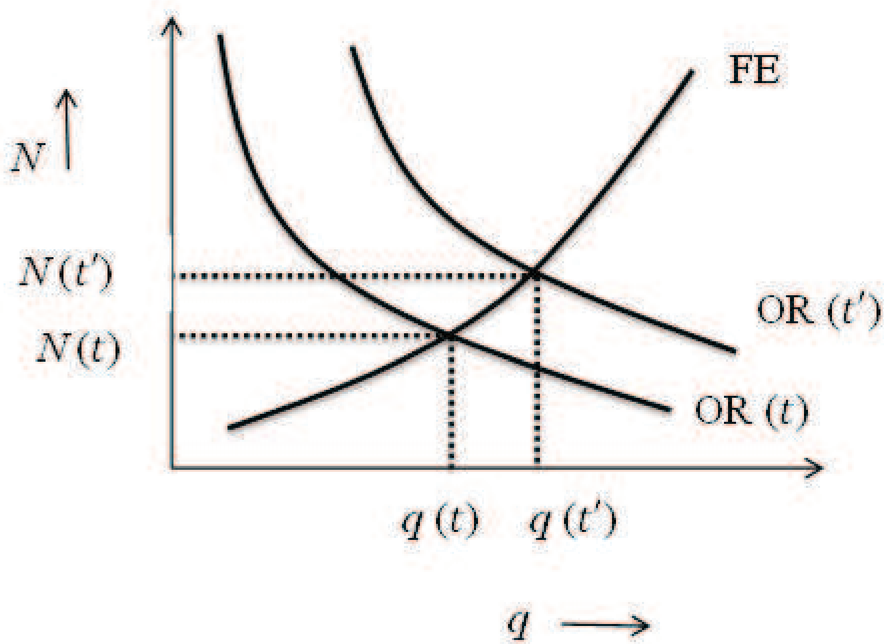

Figure 1: Increase in q and N.

As time changes from t to $t' > t$, the value of z increases, moving the (*OR*) curve to the right. At t', the input diffuses to more firms (higher N) and the quality of the input used by each firm increases.

3.2.1 Balanced Growth Path

The quality of GPT inputs like steam engines and semiconductor chips tend to improve over long periods of time (see Kyun and Pillai (2017)). For example, one measure of the quality of semiconductor chips, the number of transistors on a chip, has grown at a steady rate for the past fifty years, so much so that it has been elevated to the status of a law (Moore's law) in popular literature. This provides a motivation to look for growth paths where the quality $q(t)$ grows at a constant rate. Moreover, as we show in Proposition 1, the aggregate output Y can grow at a constant rate only along paths where the quality grows at a constant rate.

Proposition 1. *The aggregate output Y grows at a constant rate only if quality of the input q grows at a constant rate.*

Since aggregate output is $Y(t) = P(t)^{-\alpha}$ from equation (3), $Y(t)$ will grow at a constant rate only if $P(t)$ declines at a constant rate. Substituting for $A(N)$ from the *OR* condition into equation (16) gives,

$$P(t) = \frac{m_d}{\left(z(t)\frac{\varphi v}{1-\varphi}\right)^{\frac{1}{\eta-\alpha}}} q(t)^{\frac{\sigma-(\eta-1)}{\eta-\alpha}} \tag{18}$$

Since $z(t)$ grows at a constant rate, $P(t)$ will decline at a constant rate only if $q(t)$ grows at a constant rate. Hence $Y(t)$ will grow at a constant rate only if $q(t)$ grows at a constant rate.

Proposition 2. *The growth rate of quality $q(t)$ will be constant only if $a(i)$ is of the form $a(i) = Di^{-\theta}$, with $D > 0$ and $0 < \theta < \frac{1}{\eta-1}$.*[18]

Proof. First we show that if the quality growth rate $\frac{\dot{q}}{q}$ is constant, then rate of diffusion of the input, $\delta \equiv \frac{\dot{N}}{N}$, is also constant. From the *OR* condition, it can be seen that if $q(t)$ and $z(t)$ change

[18] The connection between balanced growth paths and power law distributions have been explored by many others, most notably Kortum (1997) and Jones (2005). There is a considerable empirical work that support a power law distribution of firm sizes, including Axtell (2001). There is also an extensive theoretical literature that tries to explain the fact, including Simon and Bonini (1958), Gabaix (1999) and Luttmer (2007). Gabaix (2009) provides a good review of the literature.

at constant rates, then $A(N)$ has to change at constant rate as well, i.e.,

$$A(N(t)) = A_0 e^{Mt}, \tag{19}$$

for some $M > 0$. From the ***FE*** condition, if $A(N(t))$ and $q(t)$ has to change at a constant rate, then $a(N(t))$ has to change at a constant rate as well. i.e.,

$$a(N(t)) = a_0 e^{-Jt}, \tag{20}$$

for some $J > 0$. Differentiating equation (19) with respect to time using Leibniz's rule,

$$a(N(t))^{\eta-1}\dot{N}(t) = MA_0 e^{Mt} \tag{21}$$

Substituting for $a(N(t))$ from equation (20) into equation (21) gives,

$$\dot{N}(t) = \frac{MA_0}{a_0^{\eta-1}} e^{(M+J(\eta-1))t},$$

which leads to $\dfrac{\dot{N}}{N} = M + J(\eta - 1)$, constant over time.

Next we show that it is possible to have both the quality growth rate $\dfrac{\dot{q}}{q}$ and the diffusion rate $\dfrac{\dot{N}}{N}$ constant over time only if $a(i)$ is of the form $a(i) = Di^{-\theta}$. Substituting for $A(N)^{\frac{\eta-\alpha}{\eta-1}}$ from (***FE***) equation into (***OR***) equation, we get,

$$q(t)^{\sigma} = z(t)\frac{\varphi}{1-\varphi}\frac{f}{a(N)^{\eta-1}}. \tag{22}$$

Taking logarithms and differentiating with respect to t gives,

$$\frac{\dot{q}}{q} = \frac{g}{\sigma} - \frac{\eta-1}{\sigma}\left(\frac{Na'(N)}{a(N)}\right)\frac{\dot{N}}{N} \tag{23}$$

Since $\dfrac{\dot{q}}{q}$ and $\dfrac{\dot{N}}{N}$ are constant, $\dfrac{Na'(N)}{a(N)}$ must be constant as well. Since N increases over time, $N\dfrac{a'(N)}{a(N)}$ can be constant over time only if $a(i)$ is of the form, $a(i) = Di^{-\theta}$. The restriction that $\theta < \frac{1}{\eta-1}$ is necessary to ensure that $\int_0^N a(i)^{\eta-1}$ is greater than zero. □

For the rest of the paper, we normalize $D = 1$. With $a(i) = i^{-\theta}$, we note for future use that,

$$a(N) = N^{-\theta}, \quad \frac{Na'(N)}{a(N)} = -\theta, \tag{24}$$

$$A(N) = \frac{N^{1-\tilde{\theta}}}{1-\tilde{\theta}}, \quad \frac{NA'(N)}{A(N)} = 1-\tilde{\theta}. \tag{25}$$

where we have defined $\tilde{\theta} = \theta(\eta - 1)$.

Proposition 3. *Along the unique balanced growth path, the diffusion rate is given by,*

$$\delta \equiv \frac{\dot{N}}{N} = \frac{g}{\frac{\sigma}{\alpha-1}\left(\frac{\eta-\alpha}{\eta-1}(1-\tilde{\theta}) + \tilde{\theta}(1-\frac{\alpha-1}{\sigma})\right)} \tag{26}$$

Proof. Substituting for $q(t)$ from equation (22) into the *FE* condition gives,

$$\frac{A(N)^{\frac{\eta-\alpha}{\eta-1}}}{a(N)^{(\eta-1)(1-\frac{\alpha-1}{\sigma})}} = \frac{v}{f}\left(zf\frac{\varphi}{1-\varphi}\right)^{\frac{\alpha-1}{\sigma}}. \tag{27}$$

Taking logarithms and differentiating with respect to time gives,

$$\frac{\dot{N}}{N}\left(\frac{\eta-\alpha}{\eta-1}\frac{NA'(N)}{A(N)} - (\eta-1)(1-\frac{\alpha-1}{\sigma})\frac{Na'(N)}{a(N)}\right) = \frac{\alpha-1}{\sigma}g. \tag{28}$$

Using equations (24) and (25) in equation (28) gives the solution for the diffusion rate in equation (26). □

We note, for use in Section (5), that equation (27) can used to solve for the measure of firms that have adopted at t. Substituting for $a(N)$ and $A(N)$ from equations (24) and (25) it can be seen that,

$$N(t) = \left(\left((1-\tilde{\theta})^{\frac{\eta-\alpha}{\eta-1}}\frac{v}{f}\right)^{\frac{1}{\tilde{\varphi}}}\frac{\varphi f}{1-\varphi}\right)^{\dfrac{\delta}{g}} e^{\delta t},$$

For notational convenience, we denote the measure of firms at $t = 0$ as N_0, i.e.,

$$N_0 \equiv N(0) = \left(\left((1-\tilde{\theta})^{\frac{\eta-\alpha}{\eta-1}}\frac{v}{f}\right)^{\frac{1}{\tilde{\varphi}}}\frac{\varphi f}{1-\varphi}\right)^{\dfrac{\delta}{g}}, \tag{29}$$

and hence $N(t)$ can be written as $N(t) = N_0 e^{\delta t}$.

Proposition 4. *Along the unique balanced growth path, the rate of growth of aggregate output is* $\frac{\dot{Y}}{Y} = \frac{\alpha - 1}{\alpha}\delta$*, where* δ *is the diffusion rate.*

Proof. Since all firms choose the same quality in equilibrium, the gross profit equation (7) implies,

$$\frac{\pi(i,t)}{\pi(N(t),t)} = \left(\frac{a(i)}{a(N)}\right)^{\eta-1}.$$

Since $\pi(N(t),t) = \frac{f}{1-\varphi}$ from equation (12),

$$\pi(i,t) = a(i)^{\eta-1}\frac{f}{1-\varphi}\frac{1}{a(N)^{\eta-1}}. \tag{30}$$

Given that the demand function faced by each firm is one with constant elasticity η, firm i's revenue is $\eta\pi(i,t)$. Denoting the aggregate revenue of all firms using the input by $X(t)$, we have,

$$X(t) \equiv \eta \int_{i=0}^{N(t)} \pi(i,t)di.$$

Substituting for $\pi(i,t)$ from above, taking logarithms and differentiating with respect to time, and using equation (24) and (25), we get,

$$\frac{\dot{X}}{X} = \frac{\dot{N}}{N}\left(\frac{NA'(N)}{A(N)} - (\eta-1)\frac{Na'(N)}{a(N)}\right) = \frac{\dot{N}}{N}.$$

Since $Y = P(t)^{-\alpha}$ and $X(t) = P(t)^{1-\alpha}$, the growth rate of aggregate output Y is $\frac{\dot{Y}}{Y} = \frac{\alpha}{\alpha-1}\frac{\dot{X}}{X}$. Hence,

$$\frac{\dot{Y}}{Y} = \frac{\alpha}{\alpha-1}\frac{\dot{X}}{X} = \frac{\alpha-1}{\alpha}\frac{\dot{N}}{N} = \frac{\alpha-1}{\alpha}\delta.$$

□

Now we move to the case of vertical specialization.

3.3 Vertical Specialization

To distinguish the downstream industry variables between the vertically integrated and vertically specialized cases, we will denote the variables pertaining to the downstream firms with a hat (ˆ) for the specialized case. The variables pertaining to the supplier will be subscripted with s.

We proceed under the assumption that in the vertically specialized case, all downstream firms will source from the supplier. We show in Section 4 that this is indeed the case, and partial specialization is not possible.

All downstream firms now purchase the input with the same quality $q_s(t)$ offered by the supplier, at the same price $p_s(t)$ charged by the supplier. The price charged by downstream firm i for the output good is the solution to the problem,

$$\begin{aligned} \max_{\hat{p}(i,t)} \quad & \hat{p}(i,t)\hat{y}(i,t) - p_s(t)\hat{K}(i,t) \\ s.t \quad \hat{y}(i,t) &= \hat{p}(i,t)^{-\eta}\hat{P}_t^{\eta-\alpha} \\ \hat{y}(i,t) &= a(i)q_s(t)\hat{K}(i,t). \end{aligned}$$

The optimal price is,

$$\hat{p}(i,t) = m_d \frac{p_s(t)}{a(i)q_s(t)}. \tag{31}$$

Using the demand and production functions above, the gross profit of firm i is,

$$\hat{\pi}(i,t) = \hat{p}(i,t)\hat{y}(i,t) - p_s(t)\hat{K}(i,t) = \frac{1}{\eta}\hat{p}(i,t)\hat{y}(i,t) = \frac{\hat{P}(t)^{\eta-\alpha}}{\eta}\left(\frac{a(i)q_s(t)}{m_d p_s(t)}\right)^{\eta-1}. \tag{32}$$

Further, using the solution for $\hat{p}(i,t)$ above, industry price index $\hat{P}(t)$ becomes,

$$\hat{P}(t) = \frac{m_d p_s(t)}{q_s(t)}\left(\int_{i=0}^{\hat{N}(t)} a(i)^{\eta-1} di\right)^{\frac{1}{1-\eta}} = \frac{m_d p_s(t)}{A(\hat{N})^{\frac{1}{\eta-1}} q_s(t)}.$$

where $A(\hat{N}) = \int_{i=0}^{\hat{N}(t)} a(i)^{\eta-1} di$. Using the expression for price index above, the gross profit in equation (32) can be written as,

$$\hat{\pi}(i,t) = \frac{1}{\eta(m_d p_s(t))^{\alpha-1}} \frac{a(i)^{\eta-1} q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}. \tag{33}$$

Firm i's demand for the common input is given by,

$$\hat{K}(i,t) = \frac{\hat{y}(i,t)}{a(i)q_s(t)} = \frac{1}{(m_d p_s(t))^{\alpha}} \frac{a(i)^{\eta-1} q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}. \tag{34}$$

Supplier's Problem

The demand for the input faced by the supplier, $K_s(t)$, is

$$\begin{aligned} K_s(t) &\equiv \int_{i=0}^{\hat{N}(t)} \hat{K}(i,t)di = \frac{1}{(m_d p_s(t))^{\alpha}} \frac{q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}} \int_{i=0}^{\hat{N}(t)} a(i)^{\eta-1} di \\ &= \frac{1}{(m_d p_s(t))^{\alpha}} q_s(t)^{\alpha-1} A(\hat{N})^{\frac{\alpha-1}{\eta-1}}. \end{aligned}$$

Hence the quantity of the input demanded from the supplier increases with the quality of the input, $q_s(t)$, as well as the efficiency-adjusted number of downstream firms, $A(\hat{N})$. The supplier's problem is,

$$\begin{aligned} \max_{p_s(t), q_s(t)} \quad & p_s(t)K_s(t) - K_s(t) - R_s(q_s(t), \hat{N}(t), z(t)) \\ s.t \quad K_s(t) &= \frac{1}{m_d^{\alpha}} q_s(t)^{\alpha-1} A(\hat{N})^{\frac{\alpha-1}{\eta-1}} p_s(t)^{-\alpha} \\ R_s(\hat{N}(t), q_s(t), z(t)) &= h_s(\hat{N}) \frac{q_s(t)^{\sigma}}{z(t)}. \end{aligned}$$

Faced with the constant elasticity demand curve above, the supplier's profit maximizing price is a markup of $\frac{\alpha}{\alpha-1}$ over its cost, i.e $p_s(t) = \frac{\alpha}{\alpha-1}$. Denote the supplier markup by m_s, i.e

$$p_s(t) = \frac{\alpha}{\alpha-1} \equiv m_s.$$

Hence the price charged by each downstream firm involves the standard double markup, i.e

$$\hat{p}(i,t) = \frac{m_d m_s}{q_s(t)}, \tag{35}$$

and the downstream industry price index is then,

$$\hat{P}(t) = \frac{m_d m_s}{A(\hat{N})^{\frac{1}{\eta-1}} q_s(t)}. \tag{36}$$

Hence the gross profit of firm i in equation (33) can be written as,

$$\hat{\pi}(i,t) = \frac{1}{\eta(m_d m_s)^{\alpha-1}} \frac{a(i)^{\eta-1} q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}} = \hat{v} \frac{a(i)^{\eta-1} q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}, \tag{37}$$

where we have, similar to the integrated case, defined $\hat{v}$ as the net profit (which in the case of specialization is also the gross profit) that would have accrued if there was only one downstream firm with $a = 1$ and $q = 1$, i.e.,

$$\hat{v} \equiv \frac{1}{\eta(m_d m_s)^{\alpha-1}}.$$

The gross profit obtained by the supplier is,

$$\begin{aligned}
\pi_s(q_s(t), t) &= p_s(t)X_{st} - X_{st} = \frac{1}{\alpha}p_s(t)X_{st} = \frac{1}{\alpha}m_s\frac{1}{(m_d m_s)^\alpha}q_s(t)^{\alpha-1}A(\hat{N})^{\frac{\alpha-1}{\eta-1}} \\
&= \frac{\eta-1}{\alpha}\hat{v}q_s(t)^{\alpha-1}A(\hat{N})^{\frac{\alpha-1}{\eta-1}}.
\end{aligned}$$

With the gross profit function at hand, we now turn to the R&D problem of the supplier. The supplier's R&D investment problem for period t can now be simplified to,

$$\max_{q_s(t)} \pi_s(q_s(t)) - R_s(\hat{N}(t), q_s(t), z(t)),$$

where, $R_s(q_s(t), N(t), z(t))$ and $\pi_s(q_s(t), t)$ are given above. Similar to the downstream firm's choice in the vertically integrated case, it can be seen from the first order condition to the above problem that the supplier's optimal R&D policy is to invest a fraction, $\frac{\alpha-1}{\sigma}$, of its anticipated profits into R&D. Again for notational convenience, we define a new variable, $\hat{\varphi}$, for this constant R&D to gross profit ratio of the supplier, i.e., $\hat{\varphi} \equiv \frac{\alpha-1}{\sigma}$. The only sufficient condition we require is $\alpha < 1 + \sigma$, which follows from out two earlier assumptions that $\sigma > \eta - 1$ and $\eta > \alpha$. The optimal research condition for the supplier can be expressed as,

$$q_s(t)^{\sigma-(\alpha-1)} = z(t)\frac{\eta-1}{\alpha}\frac{\hat{\varphi}\hat{v}}{h_s(\hat{N})}A(\hat{N})^{\frac{\alpha-1}{\eta-1}}. \qquad (\hat{OR})$$

Note that in contrast to the vertically integrated case, the optimal quality chosen by the supplier can be increasing or decreasing with $\hat{N}$, depending upon the ratio $\frac{A(\hat{N})^{\frac{\alpha-1}{\eta-1}}}{h_s(\hat{N})}$. Hence as long as the adaptation costs are not high, the benefits to the supplier of improving quality increases with the number of downstream firms.

Entry

As with the vertically integrated case, a firm will enter at t if the benefit from entering is higher than the benefit from waiting, i.e $\hat{\pi}(i,t) \geq f$. Since equation (37) implies that $\hat{\pi}(i,t)$ is decreasing with i for every t, there is a cutoff ability $a(i) = a(\hat{N}(t))$ for every period, such that,

$$\hat{\pi}(\hat{N}(t),t) = f \tag{38}$$

Using equation (33), the entry condition above can be written as,

$$\frac{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}{a(\hat{N})^{\eta-1}} = \frac{\hat{v}}{f} q_s(t)^{\alpha-1}. \tag{$\hat{FE}$}$$

Note that the entry condition in the specialization case is very similar to that for the integrated case, the only difference being that it reflects the absence of research done by the downstream firms and the presence of the supplier markup, incorporated in the terms v and $\hat{v}$ respectively.

Equilibrium

The equilibrium in the vertically specialized market is the set $\left(q_s(t), \hat{N}(t)\right)$ such that the optimal R&D investment condition ($\hat{OR}$) and the Free Entry condition ($\hat{FE}$) are satisfied. The following propositions characterize the equilibrium,

Proposition 5. *The diffusion rate under vertical specialization is,*

$$\hat{\delta} \equiv \frac{\dot{\hat{N}}}{\hat{N}} = \frac{g}{\frac{\sigma}{\alpha-1}\left(\frac{\eta-\alpha}{\eta-1}(1-\tilde{\theta}) + \tilde{\theta}(1-\frac{\alpha-1}{\sigma})\right) + \kappa} \tag{39}$$

Proof. Eliminating q_s from the $\hat{FE}$ and $\hat{OR}$ conditions, we get,

$$\frac{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}{a(\hat{N})^{(\eta-1)(1-\frac{\alpha-1}{\sigma})}} = \frac{\hat{v}}{f}\left(z(t)\frac{\eta-1}{\alpha}\hat{\varphi}\rho F\right)^{\frac{\alpha-1}{\sigma}}\left(\frac{A(\hat{N})}{h_s(\hat{N})}\right)^{\frac{\alpha-1}{\sigma}}. \tag{40}$$

Since we have assumed that $h(i) = a(i)^{\eta-1}$ in equation (14), we have $A(\hat{N}) = \int_{i=0}^{\hat{N}} a(i)^{\eta-1}di = \int_{i=0}^{\hat{N}} h(i)di$. Hence we can write the last term in the right hand side of the above equation as,

$$\left(\frac{A(\hat{N})}{h_s(\hat{N})}\right)^{\frac{\alpha-1}{\sigma}} = \left(\frac{\int_{i=0}^{\hat{N}} h(i)di}{h_s(\hat{N})}\right)^{\frac{\alpha-1}{\sigma}}.$$

Further, from the definition of relative R&D cost $r(.)$ in equation (1), we have,

$$r(\hat{N}) \equiv \frac{R_s(\hat{N}, q, z)}{\int_{i=0}^{\hat{N}} R(i, q, z) di} = \frac{h_s(\hat{N})}{\int_{i=0}^{\hat{N}} h(i) di}.$$

Using the two equations above, we can rewrite equation (40) as,

$$\frac{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}{a(\hat{N})^{(\eta-1)(1-\frac{\alpha-1}{\sigma})}} = \frac{\hat{v}}{f} \left(z(t) \frac{\eta-1}{\alpha} \hat{\varphi} \rho F \right)^{\frac{\alpha-1}{\sigma}} \frac{1}{r(\hat{N})^{\frac{\alpha-1}{\sigma}}}. \tag{41}$$

Taking logarithms and differentiating the above equation we get,

$$\frac{\dot{\hat{N}}}{N} \left(\frac{\eta-\alpha}{\eta-1} \frac{\hat{N} A'(\hat{N})}{A(\hat{N})} - (\eta-1)(1-\frac{\alpha-1}{\sigma}) \frac{N a'(N)}{a(N)} + \frac{\alpha-1}{\sigma} \frac{N r'(N)}{r(N)} \right) = \frac{\alpha-1}{\sigma} \frac{\dot{z}}{z} \tag{42}$$

Noting that $\kappa \equiv \frac{N r'(N)}{r(N)}$ from equation (2), and using equations (24) and (25), we get the diffusion rate under specialization as given in equation (39). □

Proposition 6. *The aggregate output growth rate under specialization is,*

$$\frac{\dot{\hat{Y}}}{\hat{Y}} = \frac{\alpha}{\alpha-1} \hat{\delta}. \tag{43}$$

Proof. The expression for gross profits in equation (33) implies that,

$$\frac{\hat{\pi}(i,t)}{\hat{\pi}(N(t),t)} = \left(\frac{a(i)}{a(\hat{N})} \right)^{\eta-1}.$$

Since $\hat{\pi}(N(t),t) = \dfrac{f}{1-\varphi}$ from equation (38),

$$\hat{\pi}(i,t) = a(i)^{\eta-1} f \frac{1}{a(\hat{N})^{\eta-1}}. \tag{44}$$

The rest of the proof is identical to that in Proposition 4 on vertical integration. □

We note, for use in Section (5), that equation (41) can be used to solve for the measure of firms, $\hat{N}(t)$, that have adopted at t. For a balanced growth path in the specialized case, equation (43) requires that the diffusion rate $\hat{\delta}$ be a constant, and hence equation (39) demands that κ be constant. For κ to be a constant, the definition in equation (2) implies that the relative R&D cost function, $r(N)$, should be

$$r(N) = \mathsf{r} N^{\kappa}, \tag{45}$$

where $\mathsf{r} = r(1)$ is the R&D cost of the supplier relative to the downstream R&D cost when $N = 1$. Substituting for $a(\hat{N})$, $A(\hat{N})$ and $r(\hat{N})$ from equations (24), (25) and (45) in equation (41), it can be seen that,

$$\hat{N}(t) = \left(\left((1-\tilde{\theta})^{\frac{\eta-\alpha}{\eta-1}} \frac{\hat{v}}{f} \right)^{\frac{1}{\hat{\varphi}}} \frac{\hat{\varphi}}{\mathsf{r}} \frac{\eta-1}{\alpha} f \right)^{\frac{\hat{\delta}}{g}} e^{\hat{\delta} t},$$

For notational convenience, we denote the measure of firms at $t = 0$ as $\hat{N}_0$, i.e.,

$$\hat{N}_0 \equiv \hat{N}(0) = \left(\left((1-\tilde{\theta})^{\frac{\eta-\alpha}{\eta-1}} \frac{\hat{v}}{f} \right)^{\frac{1}{\hat{\varphi}}} \frac{\hat{\varphi}}{\mathsf{r}} \frac{\eta-1}{\alpha} f \right)^{\frac{\hat{\delta}}{g}}, \tag{46}$$

and hence $\hat{N}(t)$ can be written as,

$$\hat{N}(t) = \hat{N}_0 e^{\hat{\delta} t}. \tag{47}$$

In the next section, we derive the condition under which the growth rate of aggregate output would be higher under specialization than integration.

4 Specialization and Output Growth Rate

Proposition 7. *The growth rate of aggregate output is higher under specialization than under integration if and only if* $\kappa < 0$.

Proof. It is clear from Propositions 4 and 6 that the growth rate of aggregate output is higher under specialization if only if the diffusion rate under specialization ($\hat{\delta}$) is higher than the diffusion rate under integration (δ). And from Propositions 3 and 5 we can see that $\hat{\delta} > \delta$ if and only if $\kappa < 0$. □

The reason that output growth rate is higher under specialization only with $\kappa < 0$ is most clearly seen by comparing the incentives to invest in R&D in the two cases. In the integrated case, for a given value of z, the quality chosen by each firm is always decreasing in the number of firms, as can be seen from equation (*OR*). In the specialized case however, whether the supplier's choice of quality increases or decreases with the number of downstream firms depends on how the R&D cost of adapting the input changes with number of firms. As can be seen from equation ($\hat{OR}$), supplier's quality choice $q_s(t)$ depends on the ratio $\frac{A(\hat{N})^{\frac{\alpha-1}{\eta-1}}}{h_s(\hat{N})}$, which can be written as $\frac{1}{r(\hat{N})A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}$. Hence if $r(.)$ is a decreasing function, suppliers choice of quality would grow faster than each downstream firm's, and consequently diffusion rate and output growth rate will also be higher under specialization.

5 Specialization as Long Run Vertical Market Structure

Until now, we have derived the paths of aggregate and firm variables taking the vertical market structure as given, either all firms are vertically integrated or all rely on the supplier. In the following propositions we answer the related question of whether or not specialized suppliers would emerge in the market. We start by defining carefully the notion of equilibrium vertical market structure.

Suppose that at time t there are $N(t)$ firms that use the general purpose input to manufacture a product, where the firms might be integrated or specialized. Let $I(t) \subseteq (0 \;\; N(t))$ be the set of vertically integrated firms, and let $S(t) \subseteq (0 \quad N(t))$ be the set of specialized firms, so that $I(t) \bigcup S(t) = (0 \;\; N(t))$.

Then $\{I(t), S(t)\}$ is an *equilibrium market structure* for t if there is no self-enforcing sub-coalition of $I(t)$ that would be better off if they sourced from the supplier, and there is no self-enforcing sub-coalition of $S(t)$ that would be better off if they had done the R&D and manufacture of the input in-house.

Proposition 8. *Partial specialization is not an equilibrium market structure for any t, other than for a trivial knife-edge case. If $\{I(t), S(t)\}$ is an equilibrium market structure, either (i)$I(t) =$*

$\varnothing, S(t) = (0\ N(t))$ or (ii) $I(t) = (0\ N(t)), S(t) = \varnothing$.

See Appendix for proof.

Next, we characterize the equilibrium market structure when $\kappa < 0$. But before proceeding to the result, we state two lemmas that are used in proving the result.

Lemma 1. *If $\kappa < 0$, then $\{I(t) = \varnothing, S(t) = (0\ N(t))\}$ is an equilibrium market structure for all $t > \tau_S$, where τ_S is given by,*

$$\tau_S = Max\{0, -\frac{1}{\kappa\hat{\delta}} \ln \left(r\hat{N_0}^{\kappa} m_s^{1+\sigma} (1-\varphi)^{\frac{1}{\varphi}-1} \right)\}, \tag{48}$$

where $\hat{\delta}$ is the diffusion rate under specialization and $\hat{N}_0$ is the number of firms that would exists under specialization at $t = 0$.

See Appendix for proof.

Lemma 2. *If $\kappa < 0$, then $\{I(t) = (0\ N(t)), S(t) = \varnothing\}$ is not an equilibrium market structure for $t > \tau_I$, where τ_I is given by*

$$\tau_I = Max\{0, -\frac{1}{\kappa\delta} \ln \left(rN_0^{\kappa} m_s^{1+\sigma} (1-\hat{\varphi})^{\frac{1}{\hat{\varphi}}-1} \right)\}. \tag{49}$$

See Appendix for proof.

Proposition 9. *If $\kappa < 0$, then specialization is the unique long run equilibrium market structure, i.e. there exists a τ such that for all $t > \tau$, $\{I(t) = \varnothing, S(t) = (0\ N(t))\}$ is the unique equilibrium market structure.*

Proof. Let $\tau = max(\tau_S, \tau_I)$, where τ_S and τ_S are as defined in Lemma 1 and Lemma 2. It follows from Proposition 8, Lemma 1 and Lemma 2 that for all $t > \tau$, specialization is the unique equilibrium market structure. □

Proposition 10. *If $\kappa = 0$, then specialization is an equilibrium market structure if only if $m_s^{1+\sigma} < \frac{1}{r}\frac{1}{(1-\varphi)^{\frac{1}{\varphi}-1}}$, a condition which does not vary over time.*

First note that if $\kappa = 0$, then equation (45) implies that $r(N) = \mathsf{r}$, independent of N. The proof is identical to the proof of Lemma 1 given in the appendix. From the proof of Lemma 1, it can be seen that the same argument applies to show that no self-enforcing coalition can make a profitable deviation from specialization to integration if $r(\hat{N}) < \frac{1}{m_s^{1+\sigma}} \frac{1}{(1-\varphi)^{\frac{1}{\varphi}-1}}$. Since $r(\hat{N}) = \mathsf{r}$ for the case $\kappa = 0$, it follows that specialization is coalition-proof if $\mathsf{r} < \frac{1}{m_s^{1+\sigma}} \frac{1}{(1-\varphi)^{\frac{1}{\varphi}-1}}$ or equivalently, if

$$m_s^{1+\sigma} < \frac{1}{\mathsf{r}} \frac{1}{(1-\varphi)^{\frac{1}{\varphi}-1}}.$$

Note that the condition for specialization in this case is similar to ones found in other models of vertical integration. Firms will rely on the supplier for the input if the supplier markup m_s is low relative to the fixed cost involved in doing the production in-house, in this case the fixed being the R&D cost as captured by the term $(1-\varphi)$ on the right hand side of the inequality above.

Proposition 11. *If $\kappa > 0$, then specialization is not a long run equilibrium market structure.*

See Appendix for proof.

In summary, Propositions 9-11 characterize the role of κ in determining the equilibrium vertical market structure. If $\kappa < 0$, then firms would prefer to source from the supplier after some point in time. If $\kappa = 0$, then firms might source from the supplier, but whether or not they do so is not influenced by the growth of the market. If specialization occurs, it will occur from time $t = 0$. Finally, with $\kappa > 0$, firms will choose the make the input in-house after some point in time.

In the next section, we apply the theory to the simplest case possible, one where all firms are identical.

6 Homogeneous Firms

Suppose that all firms have the same ability to use the general purpose input, which we normalize to one, i.e $a(i) = 1$ and $\theta = 0$. In this case, the free entry conditions (FE) and ($\hat{FE}$) hold for all firms (and not just a cutoff firm). All firms have a net present discounted value of zero. Since all firms are identical, there is no adaptation required for each firm , i.e $h(i) = a(i)^{\eta-1} = 1$. Further, since no adaptation is required, we should also have $h_s = 1$, independent of N. The relative R&D cost is then,

$$r(N) = \frac{h_s \frac{q^\sigma}{z}}{\int_{i=0}^{N} h(i)di \frac{q^\sigma}{z}} = \frac{1}{N}.$$

The coefficient of technological generality in this case is,

$$\kappa = \frac{N r'(N)}{r(N)} = -1. \tag{50}$$

Substituting $\tilde{\theta} = 0$ and $\kappa = -1$ in equations (26) and (39) gives, the diffusion rates under integration and specialization as,

$$\delta = g\frac{\phi}{\sigma}, \quad \hat{\delta} = g\frac{\frac{\phi}{\sigma}}{1-\frac{\phi}{\sigma}},$$

where $\phi = \frac{(\alpha-1)(\eta-1)}{\eta-\alpha}$.[19] The diffusion rate, and hence the aggregate output growth rate, is always higher under specialization. From equations (48) and (49) we have that,

$$\tau_S = \begin{cases} 0 & if\ \hat{N}_0 \geq \mathsf{r} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}-1}}, \\ \frac{1}{\hat{\delta}} \ln\left(\frac{\mathsf{r}}{\hat{N}_0} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}-1}}\right) & if\ \hat{N}_0 < \mathsf{r} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}-1}}. \end{cases}$$

$$\tau_I = \begin{cases} 0 & N_0 \geq \mathsf{r} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}-1}}, \\ \frac{1}{\delta} \ln\left(\frac{\mathsf{r}}{N_0} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}-1}}\right) & N_0 < \mathsf{r} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}-1}}. \end{cases}$$

where N_0 and $\hat{N}_0$ are given in equations (29) and (46),

$$N_0 = \left(\frac{v^{\frac{1}{\varphi}}}{f^{\frac{1}{\varphi}-1}} \frac{\varphi}{1-\varphi}\right)^{\frac{\phi}{\sigma}}, \quad \hat{N}_0 = \left(\frac{\hat{v}^{\frac{1}{\hat{\varphi}}}}{f^{\frac{1}{\hat{\varphi}}-1}} \frac{\hat{\varphi}}{\mathsf{r}} \frac{\eta-1}{\alpha}\right)^{\frac{\frac{\phi}{\sigma}}{1-\frac{\phi}{\sigma}}}.$$

[19]Note that $\sigma > \eta - 1$ and $\eta > \alpha$ together imply that $\frac{\phi}{\sigma} < 1$. The parameter ϕ has an economic interpretation, it is the equilibrium elasticity of diffusion with respect to quality, i.e. $\phi = \frac{\frac{dq}{dN}}{\frac{q}{N}}$. The value for ϕ above can be obtained directly by substituting $\theta = 0$ in the *FE* condition and differentiating with respect to q. See Kyun and Pillai (2017) for more details on the parameter ϕ.

For $t > \max\{\tau_S, \tau_I\}$, specialization is the unique coalition proof equilibrium market structure. Note from the equations above that if the number of firms that enter the industry at $t = 0$, given by N_0 and $\hat{N}_0$ are sufficiently high, then specialization can happen at $t = 0$.

Hence in this ideal case where the input is fully general and no adaptation is required, a specialized supplier will enter the market and the diffusion rate and aggregate output growth rate will be higher after the supplier enters the market.

7 Conclusion

We have put forward a model that illustrates a mechanism by which specialization contributes to output growth. The reason for specialization in the model is different from conventional explanations based on learning and knowledge accumulation facilitated by specialization. Instead, specialization is a market response for organizing production when there are general, reusable concepts that are useful across multiple firms, such as one would find in the production of general purpose inputs like steam engines or semiconductor chips. Specialized suppliers, in such a setting, act as substitutes for joint research consortiums. Specialization not only prevents duplication of R&D efforts by firms, but also leads to faster output growth because a common firm supplying multiple downstream firms has an incentive to invest more in R&D than each individual downstream firm.

The model developed in this paper shows that to gauge the quantitative impact of this kind of input specialization on the growth rate of output, one needs additional information on only a single parameter - the coefficient of technological generality. The parameter also plays the central role in determining whether or not specialized suppliers of a general purpose input would emerge in the market. The coefficient thus provides some clarity on the role of market growth on vertical market structure, a question that not only has long historical antecedents dating back to Adam Smith, but has also animated contemporary discussions on firm boundaries.

References

ACEMOGLU, D., V. M. CARVALHO, A. OZDAGLAR, AND A. TAHBAZ-SALEHI (2012): "The Network Origins of Aggregate Fluctuations," *Econometrica*, 80(5), 1977–2016.

AGHION, P., AND P. HOWITT (1992): "A Model of Growth through Creative Destruction," *Econometrica*, 60(2), 323–351.

——— (1998): "On the Macroeconomic Effects of Major Technological Change," in *General Purpose Technologies and Economic Growth*, ed. by E. Helpman, chap. 5, pp. 121–144. The MIT Press, Cambridge, MA.

ARORA, A., AND A. GAMBARDELLA (1994): "The changing technology of technological change: general and abstract knowledge and the division of innovative labour," *Research Policy*, 23(5), 523–532.

——— (1998): "Evolution of Industry Structure in the Chemical Industry," in *Evolution of Industry Structure in the Chemical Industry*, ed. by A. Arora, R. Landau, and N. Rosenberg. Wiley, New York.

ARROW, K. (1962): "The Economic Implications of Learning by Doing," *Review of Economic Studies*, 29(3), 155–173.

AXTELL, R. (2001): "Zipf Distribution of Firm Sizes," *Science*, 293, 1818–1820.

BECKER, G. S., AND K. M. MURPHY (1992): "The Division of Labor, Coordination Costs, and Knowledge," *The Quarterly Journal of Economics*, 107(4), 1137–60.

BERNHEIM, B. D., B. PELEG, AND M. WHINSTON (1987): "Coalition-Proof Nash Equilibria I. Concepts," *Journal of Economic Theory*, 42(1), 1–12.

BRESNAHAN, T. F. (2011): "Generality, Recombination, and Reuse," in *The Rate and Direction of Inventive Activity Revisited*, NBER Chapters, pp. 611–656. National Bureau of Economic Research, Inc.

BRESNAHAN, T. F., AND A. GAMBARDELLA (1998): "The Division of Inventive Labor and The Extent of The Market," in *General Purpose Technologies and Economic Growth*, ed. by E. Helpman, chap. 10, pp. 253–281. The MIT Press, Cambridge, MA.

BRESNAHAN, T. F., AND M. TRAJTENBERG (1995): "General purpose technologies 'Engines of growth'?," *Journal of Econometrics*, 65(1), 83–108.

CARVALHO, V. M. (2014): "From Micro to Macro via Production Networks," *Journal of Economic Perspectives*, 28(4), 23–48.

CARVALHO, V. M., AND N. VOIGTLANDER (2014): "Input Diffusion and the Evolution of Production Networks," NBER Working Papers 20025, National Bureau of Economic Research, Inc.

CHIPMAN, J. S. (1970): "External Economies of Scale and Competitive Equilibrium," *The Quarterly Journal of Economics*, 84(3), 347–385.

DIXIT, A. K., AND J. E. STIGLITZ (1977): "Monopolistic Competition and Optimum Product Diversity," *American Economic Review*, 67(3), 297–308.

GABAIX, X. (1999): "Zipf's Law for Cities: An Explanation," *The Quarterly Journal of Economics*, 114(3), 739–767.

——— (2009): "Power Laws in Economics and Finance," *Annual Review of Economics*, 1(1), 255–294.

GROSSMAN, G. M., AND E. HELPMAN (1991): "Quality Ladders in the Theory of Growth," *Review of Economic Studies*, 58(1), 43–61.

HAYEK, F. (1945): "The Use of Knowledge in Society," *American Economic Review*, 35(4), 519–530.

HELPMAN, E., AND M. TRAJTENBERG (1998): "The Diffusion of General Purpose Technologies," in *General Purpose Technologies and Economic Growth*, ed. by E. Helpman, chap. 4, pp. 85–119. The MIT Press, Cambridge, MA.

HOEFLER, D. C. (1968): "Semiconductor Family Tree," *Electronic News*.

JONES, C. I. (1995): "R&D-Based Models of Economic Growth," *Journal of Political Economy*, 103(4), 759–784.

——— (1999): "Growth: With or Without Scale Effects?," *American Economic Review*, 89(2), 139–144.

——— (2005): "The Shape of Production Functions and the Direction of Technical Change," *The Quarterly Journal of Economics*, 120(2), 517–549.

KORTUM, S. S. (1997): "Research, Patenting, and Technological Change," *Econometrica*, 65(6), 1389–1420.

KRAUS, J. (1973): "An Economic Study of The United States Semiconductor Industry," Phd dissertation, The New School for Social Research.

KYUN, K., AND U. PILLAI (2017): "A Model of Diffusion of General Purpose Technologies," *Working Paper*.

LIPSEY, R. G., K. I. CARLAW, AND C. T. BEKAR (2005): *Economic Transformations*, vol. 1. Oxford University Press, 1 edn.

LUCAS, R. J. (1988): "On the Mechanics of Economic Development," *Journal of Monetary Economics*, 22(1), 3–42.

LUTTMER, E. G. J. (2007): "Selection, Growth, and the Size Distribution of Firms," *The Quarterly Journal of Economics*, 122(3), 1103–1144.

OBERFIELD, E. (2017): "A Theory of Input-Output Architecture," *Econometria*, Forthcoming.

ROMER, P. M. (1986): "Increasing Returns and Long-run Growth," *Journal of Political Economy*, 94(5), 1002–1037.

——— (1987): "Growth Based on Increasing Returns Due to Specialization," *American Economic Review*, 77(2), 56–62.

——— (1990): "Endogenous Technological Change," *Journal of Political Economy*, 98(5), 71–102.

ROSENBERG, N. (1963): "Technological Change in the Machine Tool Industry, 1840-1910," *The Journal of Economic History*, 23(04), 414–443.

——— (1998): "Chemical Engineering as a General Purpose Technology," in *General Purpose Technologies and Economic Growth*, ed. by E. Helpman, chap. 7, p. 167. The MIT Press, Cambridge, MA.

SIMON, H., AND C. P. BONINI (1958): "The Size Distribution of Business Firms," *American Economic Review*, 48(4), 607–617.

SMITH, A. (1776): *An Inquiry into the Nature and Causes of the Wealth of Nations, edited with an Introduction, Notes, Marginal Summary and an Enlarged Index by Edwin Cannan*. Methuen, London.

STIGLER, G. J. (1951): "The Division of Labor is Limited by the Extent of the Market," *Journal of Political Economy*, 59(3), 185–193.

TILTON, J. E. (1971): *International Diffusion of Technology: Case of Semiconductors*, vol. 1. Brookings Institution, 1 edn.

UZAWA, H. (1965): "Optimum Technical Change in an Aggregative Model of Economic Growth," *International Economic Review*, 6, 18–31.

WEITZMAN, M. (1998): "Recombinant Growth," *The Quarterly Journal of Economics*, 113(2), 331–360.

A Mathematical Appendix

A.1 Proof of Proposition 8

Proposition 8 : Partial specialization is not an equilibrium market structure for any t. If $\{I(t), S(t)\}$ is an equilibrium market structure, either (i)$I(t) = \varnothing, S(t) = (0\ N(t))$ or (ii) $I(t) = (0\ N(t)), S(t) = \varnothing$.

Proof. Suppose this was not the case, and there exists an equilibrium market structure $\{I(t), S(t)\}$ such that $I(t) \neq \varnothing,\ S(t) \neq \varnothing$, for some time t. The price index $P(t)$ would now be given by,

$$P(t) = \left(\int_{i \in I(t)} p(i,t)^{1-\eta} + \int_{j \in S(t)} \hat{p}(j,t)^{1-\eta} \right)^{\frac{1}{1-\eta}}. \tag{51}$$

A firm $i \in I(t)$ faces the profit maximization problem given in Section 3.2 with $P(t)$ as given in equation (51). As shown in Section 3.2, all integrated firms will chose the same quality, with the quality and resulting profit of firm i being given by,

$$q(t) = \left(\frac{1}{\eta} \frac{\varphi z(t)}{\eta m_d^{\eta-1}} P(t)^{\eta-\alpha} \right)^{\frac{1}{\sigma-(\eta-1)}}, \tag{52}$$

$$\pi(i,t) = \frac{P(t)^{\eta-\alpha}}{\eta} \left(\frac{a(i) q(t)}{m_d} \right)^{\eta-1}. \tag{53}$$

Each firm $j \in S(t)$ faces the problem given in Section 3.3, and its profit function is as given in equation (32),

$$\hat{\pi}(j,t) = \frac{P(t)^{\eta-\alpha}}{\eta} \left(\frac{a(i) q_s(t)}{m_d p_s(t)} \right)^{\eta-1}.$$

The cut-off firm at time t can be an integrated firm or a specialized one. The proof works in either case. Suppose that the cut-off firm is an integrated one, so that its quality is $q(N,t) = q(t)$. Then the profit of the cutoff firm is as given in equation (13). Using equation (13), the gross profit of firm i above, $\pi(i,t)$, simplifies to,

$$\pi(i,t) = \left(\frac{a(i)}{a(N)} \right)^{\eta-1} \frac{f}{1-\varphi}, \tag{54}$$

and its net profit is $\left(\frac{a(i)}{a(N)}\right)^{\eta-1} \rho F$. Using the expression for the profit of the cut-off firm in equation (13), the profit of specialized firm j above, $\hat{\pi}(j,t)$ can also be written as,

$$\hat{\pi}(j,t) = \left(\frac{q_s(t)}{q(t)}\frac{a(j)}{a(N)}\frac{1}{p_s(t)}\right)^{\eta-1} \frac{f}{1-\varphi}. \tag{55}$$

If firm i had chosen to source from the supplier, then it would have obtained a profit similar to that in equation (55), i.e.,

$$\hat{\pi}(i,t) = \left(\frac{q_s(t)}{q(t)}\frac{a(i)}{a(N)}\frac{1}{p_s(t)}\right)^{\eta-1} \frac{f}{1-\varphi}. \tag{56}$$

Since firm i preferred the in-house option, it must be that $(1-\varphi)\pi(i,t) > \hat{\pi}(i,t)$, which leads to the condition,

$$q_s(t) < q(t)p_s(t)(1-\varphi)^{\frac{1}{\eta-1}}. \tag{57}$$

Similarly, if firm j had chosen the in-house option, it would have obtained a net profit of $\left(\frac{a(j)}{a(N)}\right)^{\eta-1} f$. Since j preferred to source from the supplier, it must be that $\hat{\pi}(j,t) > (1-\varphi)\pi(j,t)$, which gives,

$$q_s(t) > q(t)p_s(t)(1-\varphi)^{\frac{1}{\eta-1}}. \tag{58}$$

Equations (57) and (58) together represent a contradiction, and hence we conclude that partial specialization is not possible.

The only trivial case in which partial specialization happens is if all firms are indifferent between specialization and integration. Using the equations for $\pi(i,t)$ and $\hat{\pi}(j,t)$ above, it can can be seen that this is possible only if the condition $q_s(t) = q(t)p_s(t)(1-\varphi)^{\frac{1}{\eta-1}}$ is satisfied. We ignore this trivial knife-edge case. □

A.2 Proof of Lemma 1

Lemma 1 : If $\kappa < 0$, then $\{I(t) = \varnothing, S(t) = (0\ \hat{N}(t))\}$ is an equilibrium market structure for all $t > \tau_S$, where τ_S is given by,

$$\tau_S = -\frac{1}{\kappa\hat{\delta}} \ln\left(\mathsf{r}\,\hat{N}_0^{\kappa} m_s^{1+\sigma}(1-\varphi)^{\frac{1}{\varphi}-1}\right),$$

where $\hat{\delta}$ is the diffusion rate under specialization and $\hat{N}_0$ is the number of firms that would exists under specialization at $t = 0$.

Proof. Consider a specialization equilibrium as described in Section 3.3. Let $\hat{N}$ be the equilibrium number of downstream firms, so that $\{I(t) = \varnothing, S(t) = (0\ \hat{N}(t))\}$. Each firm faces the profit maximization problem give in Section 3.3, and its optimal choices are as derived in Section 3.3. The price charged by firm i, and the industry price index under specialization are as given in equations (35) and (36) respectively,

$$\begin{aligned}\hat{p}(i,t) &= \frac{m_d m_s}{q_s(t)}\\ \hat{P}(t) &= \frac{m_d m_s}{A(\hat{N})^{\frac{1}{\eta-1}} q_s(t)}.\end{aligned}$$

The profit made by firm i under specialization is as give in equation (37)

$$\hat{\pi}(i,t) = \frac{1}{\eta(m_d m_s)^{\alpha-1}} \frac{a(i)^{\eta-1} q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}.$$

The profit $\hat{\pi}(i,t)$ can be written in terms of $\hat{P}(t)$ using the expression for $\hat{P}(t)$ above,

$$\hat{\pi}(i,t) = \frac{\hat{P}(t)^{\eta-\alpha}}{\eta} \left(\frac{a(i) q_s(t)}{m_s m_d}\right)^{\eta-1}. \tag{59}$$

The quality offered by the supplier, $q_s(t)$, is given in equation ($\hat{OR}$). Noting that $r(\hat{N}) = \frac{R_s(\hat{N}, q_s, z)}{\int_{i=0}^{\hat{N}} R(i, q_s, z)} = \frac{h_s(\hat{N})}{\int_{i=0}^{\hat{N}} h(i) di}$, and using the expression for price index $\hat{P}$ above, equation ($\hat{OR}$) can be written as,

$$q_s(t)^{\sigma-(\eta-1)} = \frac{z(t)}{r(\hat{N})} \frac{\eta-1}{\alpha} \frac{\varphi}{\eta} \frac{1}{(m_s m_d)^{\eta-1}} \hat{P}(t)^{\eta-\alpha}. \tag{60}$$

Suppose a coalition of firms, D, decides to deviate from specialization to integration. Let D' be the set of firms that still source from the supplier, i.e $D \cup D' = (0\ \hat{N})$. In line with the notion of coalition-proofness in Bernheim, Peleg, and Whinston (1987), we will check whether a self-enforcing deviation can be profitable, assuming that the supplier firm and non-deviating downstream firms do not change their actions. Let $P_d(t)$ be the resulting price index after the deviation.

Each deviating firm in D faces the profit maximization problem in Section 3.2, with $P(t)$ replaced by $P_d(t)$. We know from Section 3.2 that deviating firm i will choose the price given in equation (6), namely, $p(i,t) = \dfrac{m_d}{a(i)q(i,t)}$. From Section 3.2 it is also clear that every deviating firm will choose the same quality, say $q_d(t)$, which can be obtained by replacing $P(t)$ in equation (*OR*) with $P_d(t)$, i.e.,

$$q_d(t)^{\sigma-(\eta-1)} = \frac{z(t)\varphi}{\eta m_d^{\eta-1}} P_d(t)^{\eta-\alpha}. \tag{61}$$

The gross profit obtained by the deviating firm i would be as given in equation (7), with $P(t)$ replaced by $P_d(t)$,

$$\pi_d(i,t) = \frac{P_d(t)^{\eta-\alpha}}{\eta} \left(\frac{a(i)q_d(t)}{m_d} \right)^{\eta-1},$$

and its net profit would be $(1-\varphi)\pi_d(i,t)$.

For the deviation to be profitable for a member of coalition D, the net profit after deviation should be higher than if it had remained as a vertically specialized firm, i.e.,we need,

$$(1-\varphi)\pi_d(i,t) > \hat{\pi}(i,t).$$

Using the expressions for $\pi_d(i,t)$ and $\hat{\pi}(i,t)$ above, the condition for deviation to be profitable is,

$$(1-\varphi)^{\frac{1}{\eta-1}} q_d(t) > \frac{q_s(t)}{m_s} \left(\frac{\hat{P}(t)}{P_d(t)} \right)^{\frac{\eta-\alpha}{\eta-1}}. \tag{62}$$

Next, we will show that if the deviating coalition is to be self-enforcing, then we should have $\frac{\hat{P}(t)}{P_d(t)} > 1$. Using the pricing equations for $p(i,t)$ and $\hat{p}(i,t)$ above, the price index $P_d(t)$ can be written as,

$$\begin{aligned} P_d(t) &= \left(\int_{i \in D} p(i,t)^{1-\eta} di + \int_{i \in D'} \hat{p}(i,t)^{1-\eta} \right)^{\frac{1}{1-\eta}} \\ &= \frac{m_s m_d}{q_s(t)} \frac{1}{\left(\left(\frac{q_d(t)}{\frac{q_s(t)}{m_s}} \right)^{\eta-1} \int_{i \in D} a(i)^{\eta-1} di + \int_{i \in D'} a(i)^{\eta-1} di \right)^{\frac{1}{\eta-1}}}. \end{aligned} \tag{63}$$

For the coalition to be self-enforcing in the sense of Bernheim, Peleg, and Whinston (1987), no sub-coalition of deviating firms should find it profitable to revert back to relying on the supplier.

Consider a sub-coalition that is arbitrarily small, so that their reverting back to specialization has negligible impact on the price index $P_d(t)$. Then the self-enforcing restriction becomes,

$$(1-\varphi)\frac{P_d(t)^{\eta-\alpha}}{\eta}\left(\frac{a(i)q_d(t)}{m_d}\right)^{\eta-1} > \frac{P_d(t)^{\eta-\alpha}}{\eta}\left(\frac{a(i)q_s(t)}{m_s m_d}\right)^{\eta-1},$$

where the left hand side is the net profit obtained by a firm as part of the deviating coalition, and the right hand side is the net profit obtained by the firm as part of the sub-coalition that reverts back to specialization. The above condition reduces to,

$$\frac{q_d(t)}{\frac{q_s(t)}{m_s}} > \frac{1}{(1-\varphi)^{\frac{1}{\eta-1}}}. \tag{64}$$

Since $(1-\varphi) < 1$, the self-enforcement condition in equation (64) implies that $\frac{q_d(t)}{\frac{q_s(t)}{m_s}} > 1$, and hence the price index $P_d(t)$ defined above should satisfy,

$$P_d(t) < \frac{m_s m_d}{q_s(t)}\frac{1}{\left(\int_{i\in D} a(i)^{\eta-1}di + \int_{i\in D'} a(i)^{\eta-1}di\right)^{\frac{1}{\eta-1}}} = \frac{m_s m_d}{q_s(t)A(\hat{N})^{\frac{1}{\eta-1}}} = \hat{P}(t), \tag{65}$$

and hence we have $\hat{P}(t) > P_d(t)$.

With $\hat{P}(t) > P_d(t)$, the condition in equation (62) implies that for deviation to be profitable it is necessary that,

$$(1-\varphi)^{\frac{1}{\eta-1}} q_d(t) > \frac{q_s(t)}{m_s}. \tag{66}$$

Finally, we will show that the condition above will be violated for $t > \tau_s$. The optimal R&D conditions in equations (60) and (61) together imply,

$$\left(\frac{q_d(t)}{q_s(t)}\right)^{\sigma-(\eta-1)} = r(\hat{N})m_s^{\eta}\left(\frac{P_d(t)}{\hat{P}_t}\right)^{\eta-\alpha}. \tag{67}$$

With $P_d(t) < \hat{P}(t)$, equation (67) implies that,

$$q_d(t) < \frac{q_s(t)}{m_s} m_s^{\frac{1+\sigma}{\sigma-(\eta-1)}} r(\hat{N})^{\frac{1}{\sigma-(\eta-1)}}. \tag{68}$$

Hence, if the relative R&D cost $r(\hat{N})$ is such that,

$$r(\hat{N}) < \frac{1}{m_s^{1+\sigma}}\frac{1}{(1-\varphi)^{\frac{\sigma-(\eta-1)}{\eta-1}}}, \tag{69}$$

then equation (68) would imply that ,

$$(1-\varphi)^{\frac{1}{\eta-1}} q_d(t) < \frac{q_s(t)}{m_s},$$

and hence the necessary condition for deviation to be profitable in equation (66) would be violated. Combining equations (47) and (45), we can write the relative R&D cost $r(.)$ as a function of t, i.e.,

$$r(\hat{N}) = \mathsf{r}\hat{N}^{\kappa} = \mathsf{r}(\hat{N}_0 e^{\hat{\delta}t})^{\kappa}.$$

Hence for all $t > \tau_S = -\frac{1}{\kappa\hat{\delta}} \ln\left(\mathsf{r}\hat{N_0}^{\kappa} m_s^{1+\sigma}(1-\varphi)^{\frac{1}{\varphi}-1}\right)$, the condition on $r(\hat{N})$ in equation (69) is satisfied, and there is no self-enforcing coalition that can make a profitable deviation from specialization. We conclude that specialization is an equilibrium market structure for $t > \tau_S$. $\square$

A.3 Proof of Lemma 2

Lemma 2 : If $\kappa < 0$, then $\{I(t) = (0\, N(t)), S(t) = \varnothing\}$ is not an equilibrium market structure for $t > \tau_I$, where τ_I is given by

$$\tau_I = -\frac{1}{\kappa d} \ln\left(\mathsf{r} N_0^{\kappa} m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}}-1}\right). \tag{70}$$

Proof. Consider a vertical integration equilibrium as given in Section 3.2. Let $N(t)$ be the equilibrium number of downstream firms, so that $\{I(t) = (0\, N(t)), S(t) = \varnothing\}$. Each firm faces the profit maximization problem given in Section 3.2 and its optimal choices are as derived in that section. Every firm would choose the same quality $q(t)$ as given in equation (*OR*), and firm i would obtain a profit given in equation (17),

$$\pi(i,t) = \frac{1}{\eta m_d^{\alpha-1}} a(i)^{\eta-1} \frac{q(t)^{\alpha-1}}{A(N)^{\frac{\eta-\alpha}{\eta-1}}}.$$

Now consider a deviating coalition of all $N(t)$ firms, who together now decide to source from the supplier. The profit of each firm in the deviating coalition will be as given in equation (37), except that $\hat{N}$ in that equation will be replaced by N, the number of firms under integration (all of whom are members of the deviating coalition), i.e.,

$$\hat{\pi}(i,t) = \frac{1}{\eta(m_d m_s)^{\alpha-1}} \frac{a(i)^{\eta-1} q_s(t)^{\alpha-1}}{A(N)^{\frac{\eta-\alpha}{\eta-1}}}$$

The deviation makes the coalition members better off if $\hat{\pi}(i,t) > (1-\varphi)\pi(i,t)$. From equations for $\pi(i,t)$ and $\hat{\pi}(i,t)$ above, it can be seen that the $\hat{\pi}(i,t) > (1-\varphi)\pi(i,t)$ if and only if,

$$\frac{q_s(t)}{m_s} > q(t)(1-\varphi)^{\frac{1}{\alpha-1}}. \tag{71}$$

The quality chosen by the supplier, for the deviating coalition of the N firms, would be as given in equation (60), except that $\hat{N}$ in the equation would be replaced by N. Equations (*OR*) and (60) would then imply that,

$$\frac{q_s(t)}{m_s} = q(t)\frac{1}{m_s^{\frac{1+\sigma}{\sigma-(\alpha-1)}}}\frac{1}{r(N)^{\frac{1}{\sigma-(\alpha-1)}}}. \tag{72}$$

Hence if the value of $r(N)$ is such that,

$$r(N) < \frac{1}{m_s^{1+\sigma}}\frac{1}{(1-\varphi)^{\frac{\sigma-(\alpha-1)}{\alpha-1}}}, \tag{73}$$

then we would have $\frac{q_s(t)}{m_s} > q(t)\frac{1}{(1-\varphi)^{\frac{\sigma-(\alpha-1)}{\alpha-1}}}$, and hence the condition for profitable deviation in equation (71) would be violated. Since $r(N) = \mathsf{r}(N_0 e^{\delta t})^{\kappa}$, for all $t > \tau_I = -\frac{1}{\kappa d}\ln\left(\mathsf{r}N_0^{\kappa}m_s^{1+\sigma}(1-\hat{\varphi})^{\frac{1}{\hat{\varphi}}-1}\right)$, the condition on $r(N)$ in equation (73) is satisfied, and hence the deviation is profitable for all members of the deviating coalition. Hence we conclude that vertical integration is not a coalition-proof equilibrium for any $t > \tau_I$. □

A.4 Proof of Proposition 11

Proposition 11 : If $\kappa > 0$, then specialization is not a long run equilibrium market structure.

Proof. With $\kappa > 0$, we have that $r(.)$ is an increasing function, with constant elasticity equal to κ. Consider an equilibrium with specialization as in Section 3.3, with $\hat{N}$ firms, all sourcing the input from the supplier of quality $q_s(t)$. The gross profit $\hat{\pi}(i,t)$ of each downstream firm would be as given in equation (37),

$$\hat{\pi}(i,t) = \frac{1}{\eta(m_d m_s)^{\alpha-1}}\frac{a(i)^{\eta-1}q_s(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}.$$

The supplier would choose quality $q_s(t)$ as given in equation (60),

$$q_s(t)^{\sigma-(\eta-1)} = \frac{\hat{P}(t)^{\eta-\alpha}}{r(\hat{N})} z(t) \frac{\eta-1}{\alpha} \varphi \frac{S}{\eta} \frac{1}{(m_s m_d)^{\eta-1}}.$$

Consider a coalition of all $\hat{N}$ firms deviating and deciding to produce the input in-house. As in Section 3.2, every deviating firm will chose the same quality $q(t)$ as given in equation (*OR*), except that N in the equation is replaced by $\hat{N}$, i.e,

$$q(t)^{\sigma-(\alpha-1)} = z(t) \frac{\varphi v}{(1-\varphi)} \frac{1}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}.$$

The deviating firms will obtain a profit $\pi(i,t)$ as given in equation (17), except that N in the equation is replaced by $\hat{N}$,

$$\pi(i,t) = \frac{S}{\eta m_d^{\alpha-1}} a(i)^{\eta-1} \frac{q(t)^{\alpha-1}}{A(\hat{N})^{\frac{\eta-\alpha}{\eta-1}}}.$$

The firms in the coalition will find the deviation to be profitable if $(1-\varphi)\pi(i,t) > \hat{\pi}(i,t)$. Using the expressions for $\pi(i,t)$ and $\hat{\pi}(i,t)$, we can see that deviation is profitable if,

$$\frac{q(t)}{\frac{q_s(t)}{m_s}} > \frac{1}{(1-\varphi)^{\frac{1}{\alpha-1}}}.$$

Using the expressions for $q_s(t)$ and $q(t)$ above, we get,

$$\frac{q(t)}{\frac{q_s(t)}{m_s}} = r(\hat{N})^{\frac{1}{\sigma-(\alpha-1)}} m_s^{\frac{1+\sigma}{\sigma-(\alpha-1)}}$$

Hence if,

$$r(\hat{N}) > \frac{1}{m_s^{1+\sigma}} \frac{1}{(1-\varphi)^{\frac{\sigma-(\alpha-1)}{\alpha-1}}}, \tag{74}$$

then we have $\frac{q(t)}{\frac{q_s(t)}{m_s}} > \frac{1}{(1-\varphi)^{\frac{1}{\alpha-1}}}$, and every deviating firm is better of being integrated than when they were sourcing from the supplier. Since $r(\hat{N}) = \mathsf{r}(\hat{N}_0 e^{\hat{d}t})^{\kappa}$ and $\kappa > 0$, for all $t > \frac{1}{\kappa \hat{d}} \ln \left(\frac{1}{\mathsf{r} \hat{N}_0^{\kappa} m_s^{1+\sigma} (1-\varphi)^{\frac{1}{\hat{\varphi}}-1}} \right)$, the condition on $r(\hat{N})$ in equation (74) is satisfied, and hence the deviation is profitable for all members of the deviating coalition. Hence we conclude that specialization is not a long run equilibrium market structure. □