How happy is a happy-face in electrostatics?

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In previous discussion, we took our first step in working out the electrostatics image of the simplest extended case: A uniformly charged rod placed near a grounded conducting sphere in a radial direction. Here, we work out the most general formation of the image in spheres.

The most general case is depicted in Fig. 1 in which we see a charge distribution within a region of space of volume V and surface S near a grounded conducting sphere. The image of an infinitesimal element of charge dq is shown by dq', therefore, the entire charge distribution forms a collective image inside the sphere within the volume V' and surface S' shown in blue.

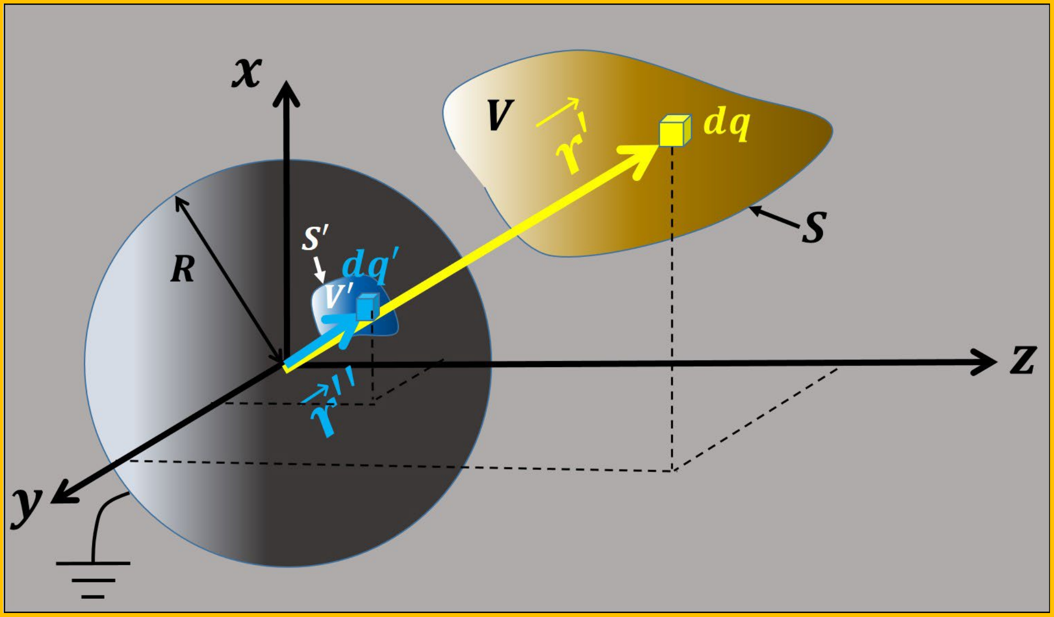

Fig. 1: A general charge distribution (brown region) near a grounded conducting sphere, and its electrostatics image inside the sphere (blue region).

Following the point charge formulas, we can write:

$$dq' = -\frac{R}{r'}dq \qquad (1)$$

$$r'' = \frac{R^2}{r'} \qquad (2)$$

Equation (1) can be used for working out the image charge density. At any θ and ϕ:

$$\rho'(r'', \theta, \phi)dv'' = -\frac{R}{r'}\rho(r', \theta, \phi)\, dv'$$

$$\rho'(r'', \theta, \phi) = -\frac{R}{r'}\ \frac{r'^2}{r''^2}\ \left|\frac{dr'}{dr''}\right|\rho(r', \theta, \phi)$$

Then, using Eq. (2):

$$\rho'(r'', \theta, \phi) = -R\ \frac{r'}{r''^2}\ \frac{R^2}{r''^2}\ \rho(r', \theta, \phi) \qquad (3)$$

If we like, we can also give the induced charge density entirely as a function of the image parameters:

$$\rho'(r'',\theta,\phi) = -\frac{R^5}{r''^5}\,\rho\left(\frac{R^2}{r''},\theta,\phi\right) \tag{4}$$

Knowing that $\vec{r}'$ and $\vec{r}''$ are in the same direction (see Fig. 1), Eq. (2) can be used to map every point in the charge distribution to its image (and vice versa), i.e. we can write:

$$\vec{r}'' = \frac{r''}{r'}\vec{r}' \tag{5}$$

consequently:

$$x'' = \frac{R^2}{r'^2}x'$$

$$y'' = \frac{R^2}{r'^2}y' \tag{6}$$

$$z'' = \frac{R^2}{r'^2}z'$$

Eq. (6) gives us a powerful mapping to work out the image shape in general. Let us consider a couple of examples:

Example 1: Horizontal rod Fig. 2 (discussed in previous poster also). In this case, $x' = y' = 0$ and $z'_L \leq z' \leq z'_R$. Therefore, Eq. (6) gives:

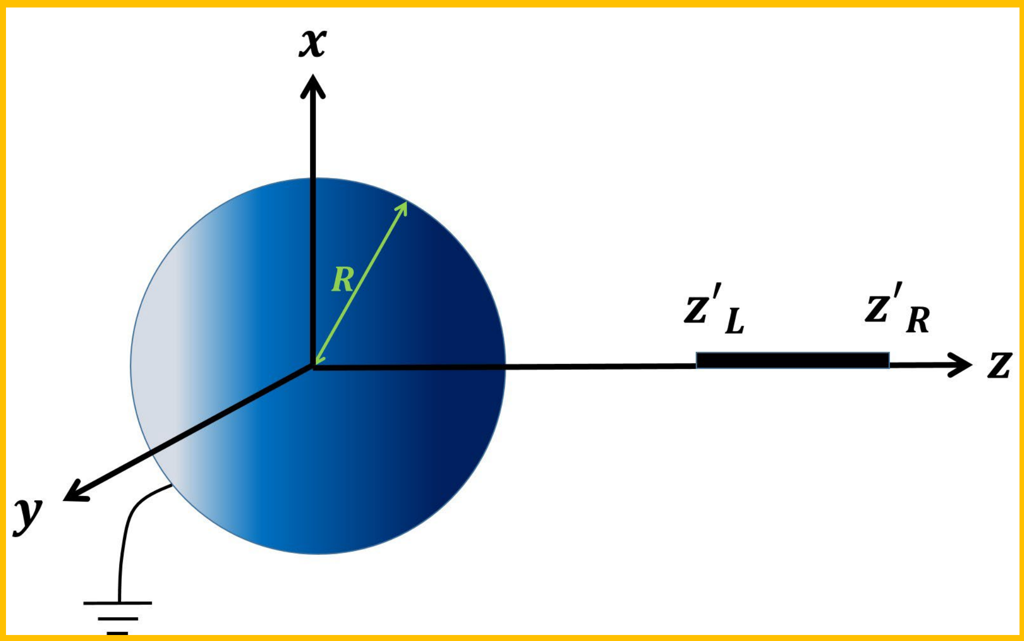

Fig. 2: A horizontal rod in front of a grounded conducting sphere.

$$x'' = y'' = 0$$

$$z'' = \frac{R^2}{z'} \qquad (7)$$

Which is what we found in previous poster as well. So, our new generalized formulas recover this old simple case!

Example 2: A vertical rod shown in Fig. 3. For this case $y' = 0, z' = d, 0 \leq x' \leq L$. Using Eq. (6) for this case, we find:

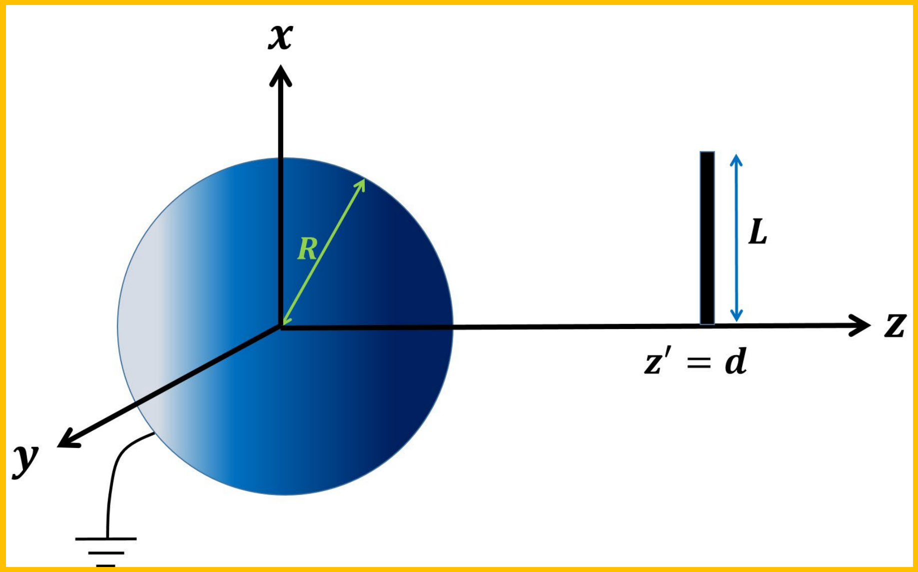

Fig. 3: A vertical charged rod in front of a grounded conducting sphere.

$$x'' = \frac{R^2}{x'^2 + d^2} x'$$

$$z'' = \frac{R^2}{x'^2 + d^2} d \qquad (8)$$

These give the parametric equation of the image which is plotted in Fig. 4. We see that, unlike the horizontal rod, the image of a vertical rod is not a straight rod, but rather is a curved rod shown in red in Fig. 4.

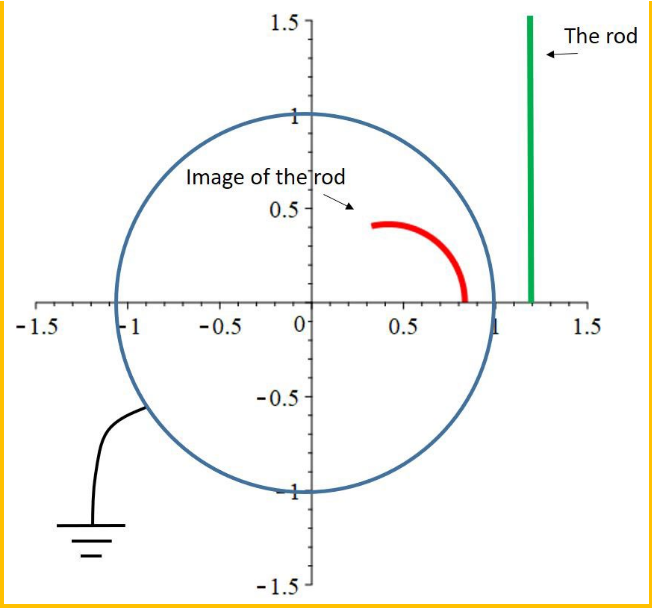

Fig. 4: A vertical rod (green) and its electrostatics image (red) in a grounded conducting sphere, computed using Eq. (6).

The Maple command for this plot (with $R = 1, d = 1.2, L = 1.5$) is:

```
plot([1.2/(xp^2 + 1.2^2), xp/(xp^2 + 1.2^2), xp = 0 .. 1.5], -1.5 .. 1.5, -1.5 .. 1.5, thickness =
5, color = red);
```

Example 3: *A happy face shown in Fig. 5 in green and its image [using (6)] is shown in red. How was this image determined?*

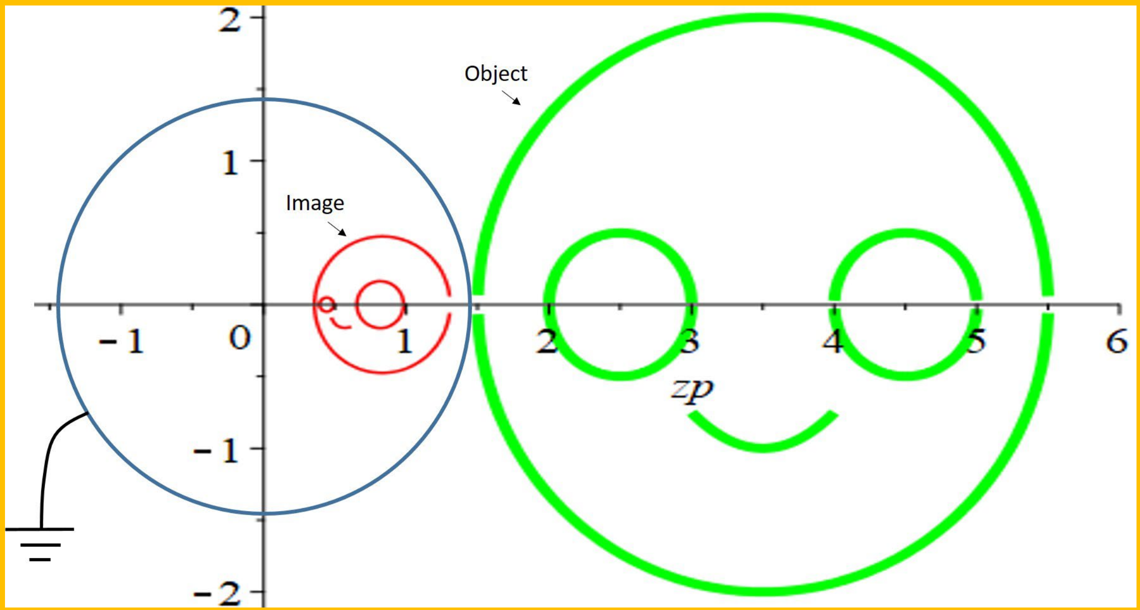

Fig. 5: A happy face (green) and its electrostatics image (red) in a grounded conducting sphere, computed using Eq. (6).

In this case $y' = 0$, and the equation of the face is:

$$x' = \pm\sqrt{R'^2 - (z' - d)^2} \tag{9}$$

The equations of the left and right eyes are:

$$x'_{left\,eye} = \pm\sqrt{R_{eye}^2 - \left(z' - d_{left\,eye}\right)^2} \tag{10}$$

$$x'_{right\,eye} = \pm\sqrt{R_{eye}^2 - \left(z' - d_{right\,eye}\right)^2} \tag{11}$$

and the equation for the smile is:

$$x'_{smile} = z'^2 - 2dz' + d^2 - \frac{R'}{2} \tag{12}$$

Eqs. (9) to (12) are substituted into Eq. (6) and the resulting parametric equations are then plotted for determination of the image. This is shown in Fig 5. The Maple code for this computation is given in the Appendix.

Since Eq. (6) gives a nonlinear map between the object and its image, some of the characteristics of the image are deformed, so the image of the happy face is transformed (but hopefully is still as happy!).

Other interesting projects:

Finding the location of the image is rather straightforward because of the mapping given by Eq. (6). However, the challenge is to:

1. Calculate the total charge on the image.
2. Calculate the electric potential produced by the object and its image.
3. Use this electric potential to calculate

the induced charge density on the surface of the sphere, and then integrate this to find the total charge and check to make sure it is equal to what we find by direct integration over the image (item 1).

These steps form excellent project ideas for motivated students who wish to deepen their knowledge of the subject and learn advanced calculus. This topic has already resulted in several very interesting projects by students in our PHY 371 (see the "Project" tab under physics webpage).

Find out more in PHY 371!

Appendix: Maple code for the "happy face"

```
R := 1.4:

Rp := 2:

d := 3.5:

zp_min := -2:

zp_max := 6:

R_eye := 0.5:

d_left_eye := 2.5:

d_right_eye := 4.5:

P_hf_up := plot(sqrt(Rp^2 - (zp - d)^2), zp = -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_hf_down := plot(-sqrt(Rp^2 - (zp - d)^2), zp = -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_le_down := plot(-sqrt(R_eye^2 - (zp - d_left_eye)^2), zp = -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_le_up := plot(sqrt(R_eye^2 - (zp - d_left_eye)^2), zp = -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_re_down := plot(-sqrt(R_eye^2 - (zp - d_right_eye)^2), zp = -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_re_up := plot(sqrt(R_eye^2 - (zp - d_right_eye)^2), zp = -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_smile := plot([zp, zp^2 - 2*d*zp + d^2 - Rp/2, zp = 3 .. 4], -2 .. 6, -3 .. 3, color = green, thickness = 5):

P_Happy_Face := display({P_hf_up, P_le_up, P_re_up, P_smile, P_hf_down, P_le_down, P_re_down}):

P_hf_up_image := plot([R^2*zp/(Rp^2 - (zp - d)^2 + zp^2), R^2*sqrt(Rp^2 - (zp - d)^2)/(Rp^2 - (zp - d)^2 + zp^2), zp = zp_min .. zp_max], -2 .. 6, -3 .. 3, color = red):

P_hf_down_image := plot([R^2*zp/(Rp^2 - (zp - d)^2 + zp^2), -R^2*sqrt(Rp^2 - (zp - d)^2)/(Rp^2 - (zp - d)^2 + zp^2), zp = zp_min .. zp_max], -2 .. 6, -3 .. 3, color = red):

P_le_up_image := plot([R^2*zp/(R_eye^2 - (zp - d_left_eye)^2 + zp^2), R^2*sqrt(R_eye^2 - (zp - d_left_eye)^2)/(R_eye^2 - (zp - d_left_eye)^2 + zp^2), zp = zp_min .. zp_max], -2 .. 6, -3 .. 3, color = red):

P_le_down_image := plot([R^2*zp/(R_eye^2 - (zp - d_left_eye)^2 + zp^2), -R^2*sqrt(R_eye^2 - (zp - d_left_eye)^2)/(R_eye^2 - (zp - d_left_eye)^2 + zp^2), zp = zp_min .. zp_max], -2 .. 6, -3 .. 3, color = red):
```

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P_re_up_image := plot([R^2*zp/(R_eye^2 - (zp - d_right_eye)^2 + zp^2), R^2*sqrt(R_eye^2 - (zp - d_right_eye)^2)/(R_eye^2 - (zp - d_right_eye)^2 + zp^2), zp = zp_min .. zp_max], -2 .. 6, -3 .. 3, color = red):

P_re_down_image := plot([R^2*zp/(R_eye^2 - (zp - d_right_eye)^2 + zp^2), -R^2*sqrt(R_eye^2 - (zp - d_right_eye)^2)/(R_eye^2 - (zp - d_right_eye)^2 + zp^2), zp = zp_min .. zp_max], -2 .. 6, -3 .. 3, color = red):

P_smile_image := plot([R^2*zp/((zp^2 - 2*d*zp + d^2 - Rp/2)^2 + zp^2), R^2*(zp^2 - 2*d*zp + d^2 - Rp/2)/((zp^2 - 2*d*zp + d^2 - Rp/2)^2 + zp^2), zp = 3 .. 4], -2 .. 6, -3 .. 3, color = red);


P_Happy_face_image := display({P_hf_down_image, P_hf_up_image, P_le_down_image, P_le_up_image, P_re_down_image, P_re_up_image, P_smile_image}):

display({P_Happy_Face, P_Happy_face_image});
```